4.1 HISTORICAL NUMERATION SYSTEMS

Basics of Numeration • Ancient Egyptian Numeration • Ancient Roman Numeration • Classical Chinese Numeration

Symbols designed to represent objects or ideas are among the oldest inventions of humans. These Indian symbols in Arizona are several hundred years old.

Basics of Numeration

The various ways of symbolizing and working with the counting numbers are called **numeration systems.** The symbols representing the numbers are called **numerals.**

Numeration systems have developed over many millennia of human history. Ancient documents provide insight into methods used by the early Sumerian peoples, the Egyptians, the Babylonians, the Greeks, the Romans, the Chinese, the Hindus, and the Mayan people, as well as others.

Keeping accounts by matching may have developed as humans established permanent settlements and began to grow crops and raise livestock. People might have kept track of the number of sheep in a flock by matching pebbles with the sheep, for example. The pebbles could then be kept as a record of the number of sheep.

A more efficient method is to keep a **tally stick.** With a tally stick, one notch or **tally** is made on a stick for each sheep. Tally marks provide a crude and inefficient numeration system. For example, the numeral for the number thirteen might be

|||||||||||||, ← 13 tally marks

which requires the recording of 13 symbols, and later interpretation requires careful counting of symbols.

Even today, tally marks are used, especially when keeping track of things that occur one or a few at a time, over space or time. To facilitate the counting of the tally, we often use a sort of "grouping" technique as we go.

卌 卌 ||| ← Numeral (tally) for 13

A long evolution of numeration systems throughout recorded history would take us from tally marks to our own modern system, the **Hindu-Arabic system,** which utilizes the set of symbols

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}.$$

That system is discussed in some detail in **Sections 4.2–4.4.**

Ancient Egyptian Numeration

Tally sticks like this one were used by the English in about 1400 A.D. to keep track of financial transactions. Each notch stands for one pound sterling.

An essential feature of all more advanced numeration systems is **grouping,** which allows for less repetition of symbols and makes numerals easier to interpret. Most historical systems, including our own, have used groups of ten, indicating that people commonly learn to count by using their fingers. The size of the groupings (again, usually ten) is called the **base** of the number system.

The ancient Egyptian system is an example of a **simple grouping system.** It utilized ten as its base, and its various symbols are shown in **Table 1** on the next page. The symbol for 1 (|) is repeated, in a tally scheme, for 2, 3, and so on up to 9. A new symbol is introduced for 10 (∩), and that symbol is repeated for 20, 30, and so on, up to 90. This pattern enabled the Egyptians to express numbers up to 9,999,999 with just the seven symbols shown in the table.

The numbers denoted by the seven Egyptian symbols are all *powers* of the base ten.

$$10^0 = 1, \quad 10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000, \quad 10^4 = 10{,}000,$$
$$10^5 = 100{,}000, \quad 10^6 = 1{,}000{,}000$$

These expressions, called *exponential expressions,* were first defined in **Section 1.1.** In the expression 10^4, for example, 10 is the *base* and 4 is the *exponent.* Recall that the exponent indicates the number of repeated factors of the base to be multiplied.

Much of our knowledge of **Egyptian mathematics** comes from the **Rhind papyrus,** from about 3800 years ago. A small portion of this papyrus, showing methods for finding the area of a triangle, is reproduced here.

Table 1 Early Egyptian Symbols

Number	Symbol	Description
1	𓏺	Stroke
10	𓎆	Heel bone
100	𓍢	Scroll
1000	𓆼	Lotus flower
10,000	𓂭	Pointing finger
100,000	𓆐	Burbot fish
1,000,000	𓁨	Astonished person

EXAMPLE 1 Interpreting an Egyptian Numeral

Write the number below in Hindu-Arabic form.

𓆐𓆐𓆼𓆼𓆼𓆼𓆼𓍢𓍢𓍢𓍢𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺𓏺𓏺

SOLUTION

Refer to **Table 1** for the values of the Egyptian symbols. Each 𓆐 represents 100,000. Therefore, two 𓆐s represent $2 \cdot 100{,}000$, or 200,000. Proceed as shown.

two	𓆐	$2 \cdot 100{,}000 =$	200,000
five	𓆼	$5 \cdot 1000 =$	5000
four	𓍢	$4 \cdot 100 =$	400
nine	𓎆	$9 \cdot 10 =$	90
seven	𓏺	$7 \cdot 1 =$	7
			205,497 ← Answer

EXAMPLE 2 Creating an Egyptian Numeral

Write 376,248 in Egyptian form.

SOLUTION

3 7 6, 2 4 8

↓ ↓ ↓ ↓ ↓ ↓

𓆐𓆐𓆐𓂭𓂭𓂭𓂭𓂭𓂭𓂭𓆼𓆼𓆼𓆼𓆼𓆼𓍢𓍢𓎆𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺 Refer to **Table 1** as needed.

An Egyptian tomb painting shows scribes tallying the count of a grain harvest. **Egyptian mathematics** was oriented more to practicality than was Greek or Babylonian mathematics, although the Egyptians did have a formula for finding the volume of a certain portion of a pyramid.

The position or order of the symbols makes no difference in a simple grouping system. Each of the numerals 𓍢𓍢𓎆𓎆𓎆𓏺𓏺𓏺𓏺, 𓏺𓏺𓏺𓏺𓎆𓎆𓎆𓍢𓍢, and 𓏺𓏺𓎆𓎆𓍢𓍢𓎆𓏺𓏺 would be interpreted as 234. In **Examples 1 and 2,** like symbols are grouped together and groups of greater-valued symbols are positioned to the left.

A simple grouping system is well suited to addition and subtraction.

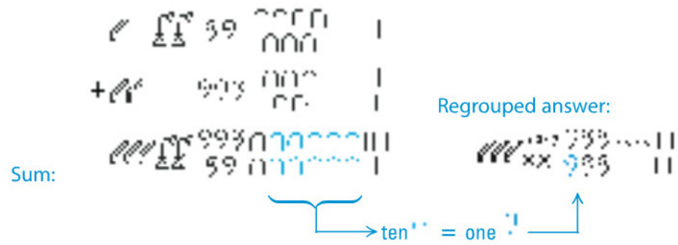
	𓆼𓆼	𓍢𓍢	𓎆𓎆𓎆	𓏺𓏺
+	𓆼	𓍢𓍢𓍢	𓎆	𓏺𓏺𓏺𓏺𓏺𓏺
Sum:	𓆼𓆼𓆼	𓍢𓍢𓍢𓍢𓍢	𓎆𓎆𓎆𓎆	𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺

We use a + sign for convenience and draw a line under the numbers being added, although the Egyptians did not do this.

Two 𓏺s plus six 𓏺s is equal to eight 𓏺s, and so on.

Archaeological investigation has provided much of what we know about the numeration systems of ancient peoples.

Sometimes regrouping, or "carrying," is needed.



Subtraction is done in much the same way, as shown in the next example.

EXAMPLE 3 Subtracting Egyptian Numerals

Work each subtraction problem.

(a)

```
  999 ∩∩ IIII
   99 ∩∩ III
 −999 ∩  IIII
```

(b)

```
  99∩∩∩∩ II
 −9  ∩∩  IIII
```

SOLUTION

(a)

```
              999 ∩∩ IIII
               99 ∩∩ III
             −999 ∩  IIII
Difference:    99 ∩∩∩ III
```

As with addition, work from right to left and subtract.

(b) To subtract four I s from two I s, "borrow" one heel bone, which is equivalent to ten I s. Finish the problem after writing ten additional I s on the right.

```
Regrouped:   99 ∩∩∩ IIIIII
                    IIIIII
             −9  ∩∩  IIII
Difference:   9  ∩   IIIIIIII
```

one ∩ = ten Is

A procedure such as those described above is called an **algorithm:** a rule or method for working a problem. The Egyptians used an interesting algorithm for multiplication that requires only an ability to add and to double numbers, as shown in **Example 4.** For convenience, this example uses our symbols rather than theirs.

EXAMPLE 4 Using the Egyptian Multiplication Algorithm

A rectangular room in an archaeological excavation measures 19 cubits by 70 cubits. (A cubit, based on the length of the forearm, from the elbow to the tip of the middle finger, was approximately 18 inches.) Find the area of the room.

SOLUTION

Multiply the width and length to find the area of a rectangle. Build two columns of numbers as shown at the top of the next page. Start the first column with 1, the second with 70. Each column is built downward by doubling the number above. Keep going until the first column contains numbers that can be added to equal 19. Then add the corresponding numbers from the second column.

	→ 1	70 ←	
	→ 2	140 ←	
1 + 2 + 16 = 19	4	280	70 + 140 + 1120 = 1330
	8	560	
	→ 16	1120 ←	

Thus $19 \cdot 70 = 1330$, and the area of the given room is 1330 square cubits.

Ancient Roman Numeration

Roman numerals are still used today, mainly for decorative purposes, on clock faces, for heading numbers in outlines, chapter numbers in books, copyright dates of movies, and so on. The base is again 10, with distinct symbols for 1, 10, 100, and 1000. The Romans, however, deviated from pure simple grouping in several ways. For the symbols and some examples, see **Tables 2 and 3**, respectively.

Table 2 Roman Symbols

Number	Symbol
1	I
5	V
10	X
50	L
100	C
500	D
1000	M

Table 3 Selected Roman Numerals

Number	Numeral
6	VI
12	XII
19	XIX
30	XXX
49	XLIX
85	LXXXV
25,040	$\overline{\text{XXV}}\,\text{XL}$
35,000	$\overline{\text{XXXV}}$
5,105,004	$\overline{\overline{\text{V}}}\,\overline{\text{CV}}\,\text{IV}$
7,000,000	$\overline{\overline{\text{VII}}}$

Special Features of the Roman System

1. In addition to symbols for 1, 10, 100, and 1000, "extra" symbols denote 5, 50, and 500. This allows less symbol repetition within a numeral. It is like a secondary base 5 grouping functioning within the base 10 simple grouping.
2. A ***subtractive feature*** was introduced, whereby a smaller-valued symbol, placed immediately to the left of one of larger value, meant to subtract. Thus IV = 4, while VI = 6. Only certain combinations were used in this way:
   (a) I preceded only V or X.
   (b) X preceded only L or C.
   (c) C preceded only D or M.
3. A ***multiplicative feature,*** rather than more symbols, allowed for larger numbers:
   (a) A bar over a numeral meant to multiply by 1000.
   (b) A double bar meant to multiply by 1000^2, that is, by 1,000,000.

Adding and subtracting with Roman numerals is very similar to the Egyptian method, except that the subtractive feature of the Roman system sometimes makes the processes more involved. With Roman numerals we cannot add IV and VII to get the sum VVIII by simply combining like symbols. (Even XIII would be incorrect.) The safest method is to rewrite IV as IIII, then add IIII and VII, getting VIIIIII. We convert this to VVI, and then to XI by regrouping. Subtraction, which is similar, is shown in the following example.

EXAMPLE 5 Subtracting Roman Numerals

Thomas DiGiano, a Roman official, has 26 servants. If, on a given Saturday, he has excused 14 of them to attend a Lucky Lyres concert at the Forum, how many are still at home to serve the banquet?

SOLUTION

To find the answer, we subtract XIV from XXVI. Set up the problem in terms of simple grouping numerals (that is, XIV is rewritten as XIIII):

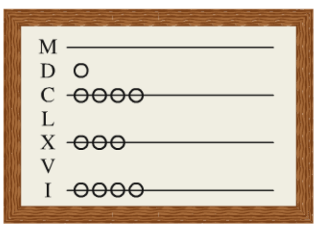
Problem:	XXVI − XIV	Problem restated without subtractive notation:	XXVI − XIIII

Regrouped:

```
XXIIIIII
 − XIIII
     XII  ← Answer
```

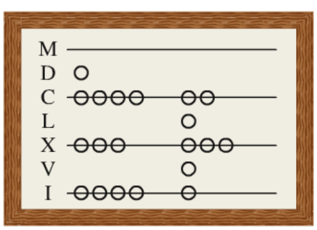
Since four **I**s cannot be subtracted from one **I**, we have "borrowed" in the top numeral, writing XXVI as XXIIIIII. The subtraction can then be carried out. Thomas has 12 servants home for the banquet.

Computation, in early forms, was often aided by mechanical devices just as it is today. The Roman merchants, in particular, did their figuring on a counting board, or **counter,** on which lines or grooves represented 1s, 10s, 100s, etc., and on which the spaces between the lines represented 5s, 50s, 500s, and so on. Discs or beads (called *calculi*, the word for "pebbles") were positioned on the board to denote numbers, and *calculations* were carried out by moving the discs around and simplifying.



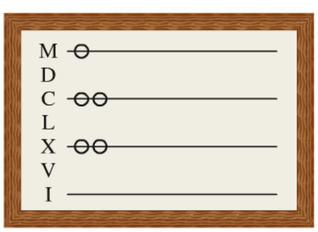
934

Figure 1



934 + 286

Figure 2



1220

Figure 3

EXAMPLE 6 Adding on a Roman Counting Board

A Roman merchant wants to calculate the sum 934 + 286. Use counting boards to carry out the following steps.

(a) Represent the first number, 934.

(b) Represent the second number, 286, beside the first.

(c) Represent the sum, in simplified form.

SOLUTION

(a) See **Figure 1**. **(b)** See **Figure 2**.

(c) See **Figure 3**. The simplified answer is MCCXX, or 1220. In the process of simplification, five discs on the bottom line were replaced by a single disc in the V space. This made two Vs that were replaced by an additional disc on the X line. Five of those on the X line were then replaced by one in the L space, and this process continued until the disc on the M line finally appeared.

Classical Chinese Numeration

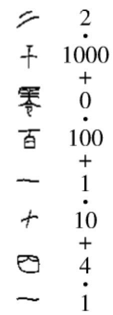
The preceding examples show that simple grouping, although an improvement over tallying, still requires considerable repetition of symbols. To denote 90, for example, the ancient Egyptian system must utilize nine ∩s: ∩∩∩∩∩ ∩∩∩∩. If an additional symbol (a "multiplier") was introduced to represent nine, say "9," then 90 could be denoted 9 ∩. All possible numbers of repetitions of powers of the base could be handled by introducing a separate multiplier symbol for each counting number less than the base.

Table 4 Chinese Symbols

Number	Symbol
1	一
2	二
3	三
4	四
5	五
6	六
7	七
8	八
9	九
10	十
100	百
1000	千
0	零

Just such a system was developed many years ago in China. We show the predominant Chinese version, which used the symbols shown in **Table 4**. We call this type of system a **multiplicative grouping system.** In general, a numeral in such a system would contain pairs of symbols, each pair containing a multiplier (with some counting number value less than the base) and then a power of the base. The Chinese numerals are read from top to bottom rather than from left to right.

If the Chinese system were *pure* multiplicative grouping, the number 2014 would be denoted as shown in **Figure 4**. But three special features of the system show that they had started to move beyond multiplicative grouping toward something more efficient.

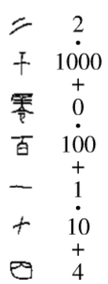


2014 in pure multiplicative grouping

Figure 4

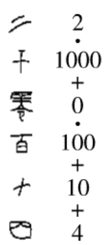
Special Features of the Chinese System

1. A single symbol, rather than a pair, denotes the number of 1s. The multiplier (1, 2, 3, 4, . . . , or 9) is written, but the power of the base (10^0) is omitted. See **Figure 5** (and also **Examples 7(a), (b), and (c)**).
2. In the 10s pair, if the multiplier is 1 it is omitted. See **Figure 6** (and **Example 8(a)**).
3. When a particular power of the base is totally missing, the omission is denoted with the zero symbol. See **Figure 7** (and **Examples 7(b) and 8(b)**). If two or more consecutive powers are missing, just one zero symbol denotes the total omission. (See **Example 7(c)**.) The omission of 1s and 10s and any other powers occurring at the extreme bottom of a numeral need not be denoted at all. (See **Example 7(d)**.)



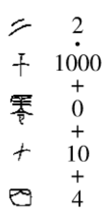
2014 with feature 1

Figure 5



2014 with features 1 and 2

Figure 6



2014 with features 1, 2, and 3

Figure 7

Note that, for clarification in the following examples, we have emphasized the grouping into pairs by spacing and by colored braces. These features were *not* part of the actual numerals in practice.

EXAMPLE 7 Interpreting Chinese Numerals

Interpret each Chinese numeral.

(a) 三 千 一 百 六 十 四 (read top to bottom)

(b) 七 百 零 三

(c) 五 千 零 九

(d) 四 千 二 百

SOLUTION

(a)

三 千 } 3 · 1000 = 3000

一 百 } 1 · 100 = 100

六 十 } 6 · 10 = 60

四 4(· 1) = 4

Total: 3164

(b)

七 百 } 7 · 100 = 700

零 0(· 10) = 00

三 3(· 1) = 3

Total: 703

This illustration is of a **quipu.** In *Ethnomathematics: A Multicultural View of Mathematical Ideas,* Marcia Ascher writes:

> *A quipu is an assemblage of colored knotted cotton cords. Cotton cordage and cloth were of unparalleled importance in Inca culture. The colors of the cords, the way the cords are connected, the relative placement of the cords, the spaces between the cords, the types of knots on the individual cords, and the relative placement of the knots are all part of the logical-numerical recording.*

(c)

五 千 } $5 \cdot 1000 = 5000$

零 } $0(\cdot 100) = 000$, $0(\cdot 10) = 00$

九 $9(\cdot 1) = 9$

Total: 5009

(d)

四 千 } $4 \cdot 1000 = 4000$

二 百 } $2 \cdot 100 = 200$

Total: 4200

EXAMPLE 8 Creating Chinese Numerals

Write a Chinese numeral for each number.

(a) 614 **(b)** 5090

SOLUTION

(a) The number 614 is made up of six 100s, one 10, and four 1s, as depicted at the right.

$6 \cdot 100$: 六 百

$(1 \cdot)10$: 十

$4(\cdot 1)$: 四

(b) The number 5090 consists of five 1000s, no 100s, and nine 10s (no 1s).

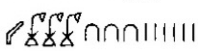
$5 \cdot 1000$: 五 千

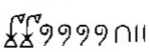
$0(\cdot 100)$: 零

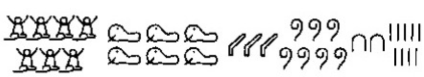
$9 \cdot 10$: 九 十

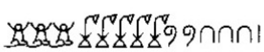
4.1 EXERCISES

Convert each Egyptian numeral to Hindu-Arabic form.

1. 

2. 

3. 

4. 

Convert each Hindu-Arabic numeral to Egyptian form.

5. 23,145 **6.** 427

7. 8,657,000 **8.** 306,090

Chapter 1 of the book of Numbers in the Bible describes a census of the draft-eligible men of Israel after Moses led them out of Egypt into the Desert of Sinai, about 1450 B.C. Write an Egyptian numeral for the number of available men from each tribe listed.

9. 59,300 from the tribe of Simeon

10. 46,500 from the tribe of Reuben

11. 74,600 from the tribe of Judah

12. 45,650 from the tribe of Gad

13. 62,700 from the tribe of Dan

14. 54,400 from the tribe of Issachar

Convert each Roman numeral to Hindu-Arabic form.

15. CLXXXII **16.** MDXCVII

17. $\overline{\text{XIV}}$ **18.** $\overline{\overline{\text{V}}}\overline{\text{CXXI}}\text{D}$

Convert each Hindu-Arabic numeral to Roman form.

19. 2861 **20.** 749

21. 25,619 **22.** 6,402,524

Convert each Chinese numeral to Hindu-Arabic form.

23. 九 百 三 十 五

24. 二 百 四 十 六

25. 三 千 零 七

26. 四 千 九 百 零 二

Convert each Hindu-Arabic numeral to Chinese form.

27. 960 **28.** 63 **29.** 7012 **30.** 2416

Though Chinese art forms began before written history, their highest development was achieved during four particular dynasties. Write traditional Chinese numerals for the beginning and ending dates of each dynasty listed.

31. Ming (1368 to 1644)

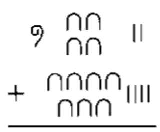
32. Sung (960 to 1279)

33. T'ang (618 to 907)

34. Han (202 B.C. to A.D. 220)

Work each addition or subtraction problem, using regrouping as necessary. Convert each answer to Hindu-Arabic form.

35.



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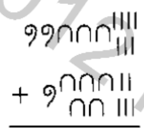

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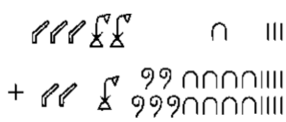

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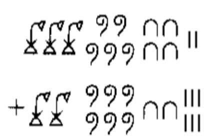

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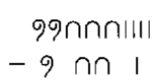

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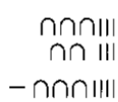

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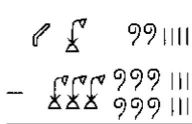
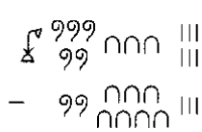


42.



Use the Egyptian algorithm to find each product.

43. $32 \cdot 47$

44. $29 \cdot 75$

45. $64 \cdot 127$

46. $52 \cdot 131$

In Exercises 47 and 48, convert all numbers to Egyptian numerals. Multiply using the Egyptian algorithm, and add using the Egyptian symbols. Give the final answer using a Hindu-Arabic numeral.

47. ***Value of a Biblical Treasure*** The book of Ezra in the Bible describes the return of the exiles to Jerusalem. When they rebuilt the temple, the King of Persia gave them the following items: thirty golden basins, a thousand silver basins, four hundred ten silver bowls, and thirty golden bowls. Find the total value of this treasure, if each gold basin is worth 3000 shekels, each silver basin is worth 500 shekels, each silver bowl is worth 50 shekels, and each golden bowl is worth 400 shekels.

48. ***Total Bill for King Solomon*** King Solomon told the King of Tyre (now Lebanon) that Solomon needed the best cedar for his temple, and that he would "pay you for your men whatever sum you fix." Find the total bill to Solomon if the King of Tyre used the following numbers of men: 5500 tree cutters at two shekels per week each, for a total of seven weeks; 4600 sawers of wood at three shekels per week each, for a total of 32 weeks; and 900 sailors at one shekel per week each, for a total of 16 weeks.

Explain why each step would be an improvement in the development of numeration systems.

49. progressing from carrying groups of pebbles to making tally marks on a stick

50. progressing from tallying to simple grouping

51. utilizing a subtractive technique within simple grouping, as the Romans did

52. progressing from simple grouping to multiplicative grouping

Recall that the ancient Egyptian system described in this section was simple grouping, used a base of ten, and contained seven distinct symbols. The largest number expressible in that system is 9,999,999. Identify the largest number expressible in each of the following simple grouping systems. (In Exercises 57–60, d can be any counting number.)

53. base ten, five distinct symbols

54. base ten, ten distinct symbols

55. base five, five distinct symbols

56. base five, ten distinct symbols

57. base ten, d distinct symbols

58. base five, d distinct symbols

59. base seven, d distinct symbols

60. base b, d distinct symbols (where b is any counting number 2 or greater)

61. The Chinese system presented in the text has symbols for 1 through 9, and also for 10, 100, and 1000. What is the greatest number expressible in that system?

62. The Chinese system did eventually adopt two additional symbols, for 10,000 and 100,000. What greatest number could then be expressed?

63. If the first (least-valued) six symbols of the Roman system are arranged from greatest value to least, left to right, what famous number is denoted?

64. The number in **Exercise 63** is denoted with six symbols as a Roman numeral. How many symbols would it require as
(a) a Chinese numeral?
(b) an Egyptian numeral?

4.2 MORE HISTORICAL NUMERATION SYSTEMS

Basics of Positional Numeration

A simple grouping system relies on repetition of symbols to denote the number of each power of the base. A multiplicative grouping system uses multipliers in place of repetition, which is more efficient. The ultimate in efficiency is attained with a **positional system** in which only multipliers are used. The various powers of the base require no separate symbols, because the power associated with each multiplier can be understood by the position that the multiplier occupies in the numeral.

If the Chinese system had evolved into a positional system, then the numeral for 7482 could be written

七
四
八
二 rather than 七
千
四
百
八
十
二 .

In the positional version on the left, the lowest symbol is understood to represent two 1s (10^0), the next one up denotes eight 10s (10^1), then four 100s (10^2), and finally seven 1000s (10^3). Each symbol in a numeral now has both a *face value,* associated with that particular symbol (the multiplier value), and a *place value* (a power of the base), associated with the place, or position, occupied by the symbol.

Positional Numeration

In a positional numeral, each symbol (called a **digit**) conveys two things:

1. **face value**—the inherent value of the symbol
2. **place value**—the power of the base that is associated with the position that the digit occupies in the numeral.

Hindu-Arabic Numeration

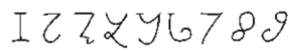
The place values in a Hindu-Arabic numeral, from right to left, are 1, 10, 100, 1000, and so on. The three 4s in the number 46,424 all have the same face value but different place values. The first 4, on the left, denotes four 10,000s, the next one denotes four 100s, and the one on the right denotes four 1s. Place values (in base ten) are named as shown here.

Billions,	Hundred millions	Ten millions	Millions,	Hundred thousands	Ten thousands	Thousands,	Hundreds	Tens	Units	Decimal point
8,	3	2	1,	4	5	6,	7	9	5	.

This numeral is read as eight billion, three hundred twenty-one million, four hundred fifty-six thousand, seven hundred ninety-five.

To work successfully, a positional system must have a symbol for zero to serve as a **placeholder** in case one or more powers of the base are not needed. Because of this requirement, some early numeration systems took a long time to evolve to a positional form, or never did. Although the traditional Chinese system does utilize a zero symbol, it never did incorporate all the features of a positional system, but remained essentially a multiplicative grouping system.

The one numeration system that did achieve the maximum efficiency of positional form is our own system, the **Hindu-Arabic** system. Its symbols have been traced to the Hindus of 200 B.C. They were picked up by the Arabs and eventually transmitted to Spain, where a late tenth-century version appeared like this:



The earliest stages of the system evolved under the influence of navigational, trade, engineering, and military requirements. And in early modern times, the advance of astronomy and other sciences led to a structure well suited to fast and accurate computation.

The purely positional form that the system finally assumed was introduced to the West by Leonardo Fibonacci of Pisa (1170–1250) early in the thirteenth century, but widespread acceptance of standardized symbols and form was not achieved until the invention of printing during the fifteenth century. Since that time, no better system of numeration has been devised, and the positional base ten Hindu-Arabic system is commonly used around the world today.

The Hindu-Arabic system and notation will be investigated further in **Sections 4.3 and 4.4**. The systems we consider next, the Babylonian and the Mayan, achieved the main ideas of positional numeration without fully developing those ideas.

Table 5 Babylonian Symbols

Number	Symbol
1	
10	

Babylonian Numeration

The Babylonians used a base of 60 in their system. Because of this, in theory they would then need distinct symbols for numbers from 1 through 59 (just as we have symbols for 1 through 9). However, the Babylonian method of writing on clay with wedge-shaped sticks gave rise to only *two* symbols, as shown in **Table 5**. The number 47 would be written

 or . The number 47

Since the Babylonian system had base 60, the "digit" on the right in a multi-digit number represented the number of 1s, with the second "digit" from the right giving the number of 60s. The third digit would give the number of 3600s ($60 \cdot 60 = 3600$), and so on.

Special Features of the Babylonian System

1. Rather than using distinct symbols for each number less than the base (60), the Babylonians expressed face values in base 10 simple grouping, using only the two symbols

    for 10 and for 1.

   The system is, therefore, base 10 simple grouping *within* base 60 positional.
2. The earliest Babylonian system lacked a place holder symbol (zero), so missing powers of the base were difficult to express. Blank spaces within a numeral would be open to misinterpretation.

EXAMPLE 1 Converting Babylonian Numerals to Hindu-Arabic

Convert each Babylonian numeral to Hindu-Arabic form.

(a)  **(b)**  **(c)** 

SOLUTION

(a) Here we have five 10s and three 1s.

$$\begin{aligned} 5 \cdot 10 &= 50 \\ 3 \cdot 1 &= 3 \\ \hline &\ 53 \leftarrow \text{Answer} \end{aligned}$$

(b) This "two-digit" Babylonian number represents twenty-two 1s and fifty-eight 60s.

$$\begin{aligned} 22 \cdot 1 &= 22 \\ 58 \cdot 60 &= 3480 \\ \hline &\ 3502 \leftarrow \text{Answer} \end{aligned}$$

(c) Here we have a three-digit number.

$$\begin{aligned} 36 \cdot 1 &= 36 \\ 11 \cdot 60 &= 660 \\ 25 \cdot 3600 &= 90{,}000 \\ \hline &\ 90{,}696 \leftarrow \text{Answer} \end{aligned}$$

▮▮▮

EXAMPLE 2 Converting Hindu-Arabic Numerals to Babylonian

Convert each Hindu-Arabic numeral to Babylonian form.

(a) 733 **(b)** 75,904 **(c)** 43,233

SOLUTION

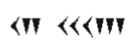
(a) To write 733 in Babylonian, we will need some 60s and some 1s. Divide 60 into 733. The quotient is 12, with a remainder of 13. Thus we need twelve 60s and thirteen 1s.

 ← 733

(b) For 75,904, we need some 3600s, as well as some 60s and some 1s. Divide 75,904 by 3600. The answer is 21, with a remainder of 304. Divide 304 by 60. The quotient is 5, with a remainder of 4.

 ← 75,904

(c) Divide 43,233 by 3600. The answer is 12, with a remainder of 33. We need no 60s here. In a system such as ours we would use a 0 to show that no 60s are needed. Since the early Babylonians had no such symbol, they merely left a space.

 ← 43,233

▮▮▮

Example 2(c) illustrates the problem presented by the lack of a symbol for zero. In our system we know that 202 is not the same as 2002 or 20,002. The lack of a zero symbol was a major difficulty with the very early Babylonian system. A symbol for zero was introduced about 300 B.C.

Mayan Numeration

The Mayan Indians of Central America and Mexico also used what is basically a positional system. Like the Babylonians, the Mayans did not use base 10—they used base 20, with a twist. In a true base 20 system, the digits would represent 1s, 20s, $20 \cdot 20 =$ 400s, $20 \cdot 400 = 8000$s, and so on. The Mayans used 1s, 20s, $18 \cdot 20 = 360$s, $20 \cdot 360 = 7200$s, and so on. It is possible that they multiplied 20 by 18 (instead of 20) since $18 \cdot 20$ is close to the number of days in a year, convenient for astronomy. The symbols of the Mayan system are shown in **Table 6**.

Table 6 Mayan Symbols

Number	Symbol
0	
1	·
5	—

The Mayans were one of the first civilizations to invent a placeholder. They had a zero symbol many hundreds of years before it reached western Europe. Mayan numerals are written from top to bottom, just as in the classical Chinese system.

Special Features of the Mayan System

1. Rather than using distinct symbols for each number less than the base (20), the Mayans expressed face values in base 5 simple grouping, using only the two symbols — for 5 and · for 1. The system is, therefore, base 5 simple grouping *within* base 20 positional.
2. Place values in base 20 would normally be

$$1, \quad 20, \quad 20^2 = 400, \quad 20^3 = 8000,$$
$$20^4 = 160{,}000, \quad \text{and so on.}$$

However, the Mayans multiplied by 18 rather than 20 in just one case, so the place values are

$$1, \quad 20, \quad 20 \cdot 18 = 360, \quad 360 \cdot 20 = 7200,$$
$$7200 \cdot 20 = 144{,}000, \quad \text{and so on.}$$

EXAMPLE 3 Converting Mayan Numerals to Hindu-Arabic

Convert each Mayan numeral to Hindu-Arabic form.

(a)  **(b)** 

SOLUTION

(a) The top group of symbols represents twelve 20s, while the bottom group represents nine 1s.

$$\begin{aligned} 12 \cdot 20 &= 240 \\ 9 \cdot 1 &= \underline{9} \\ &\ 249 \leftarrow \text{Answer} \end{aligned}$$

(b)

$$\begin{aligned} 8 \cdot 360 &= 2880 \\ 0 \cdot 20 &= 0 \\ 15 \cdot 1 &= \underline{15} \\ &\ 2895 \leftarrow \text{Answer} \end{aligned}$$

EXAMPLE 4 Converting Hindu-Arabic Numerals to Mayan

Convert each Hindu-Arabic numeral to Mayan form.

(a) 277 **(b)** 1238

SOLUTION

(a) The number 277 requires thirteen 20s (divide 277 by 20) and seventeen 1s.

 ← 277

(b) Divide 1238 by 360. The quotient is 3, with remainder 158. Divide 158 by 20. The quotient is 7, with remainder 18. Thus we need three 360s, seven 20s, and eighteen 1s.

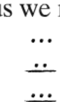
 ← 1238

Table 7 Greek Symbols

Number	Symbol
1	α
2	β
3	γ
4	δ
5	ϵ
6	ς
7	ζ
8	η
9	θ
10	ι
20	κ
30	λ
40	μ
50	ν
60	ξ
70	ο
80	π
90	ϙ
100	ρ
200	σ
300	τ
400	υ
500	ϕ
600	χ
700	ψ
800	ω
900	ϡ

Greek Numeration

The classical Greeks of Ionia assigned values to the 24 letters of their ordinary alphabet, together with three obsolete Phoenician letters (the digamma ς for 6, the koppa ϙ for 90, and the sampi ϡ for 900). See **Table 7**. This scheme, usually called a **ciphered system,** makes all counting numbers less than 1000 easily represented. It avoids repetitions of symbols but requires vast multiplication tables for 27 distinct symbols. Computation would be very burdensome. The base is 10, but the system is quite different than simple grouping, multiplicative grouping, or positional.

EXAMPLE 5 Converting Greek Numerals to Hindu-Arabic

Convert each Greek numeral to Hindu-Arabic form.

(a) λα **(b)** τξϵ **(c)** ϡϙθ **(d)** χδ

SOLUTION

(a) 31 **(b)** 365 **(c)** 999 **(d)** 604

For numbers larger than 999, the Greeks introduced two additional techniques.

Special Features of the Greek System

1. Multiples of 1000 (up to 9000) are indicated with a small stroke next to a units symbol. For example, 9000 would be denoted ͵θ.
2. Multiples of 10,000 are indicated by the letter M (from the word *myriad*, meaning ten thousand) with the multiple (a units symbol) shown above the M. The number 50,000 would be denoted $\overset{\epsilon}{M}$.

EXAMPLE 6 Converting Hindu-Arabic Numerals to Greek

Convert each Hindu-Arabic numeral to Greek form.

(a) 3000 **(b)** 40,000 **(c)** 7694 **(d)** 88,888

SOLUTION

(a) ͵γ **(b)** $\overset{\delta}{M}$ **(c)** ͵ζχϙδ **(d)** $\overset{\eta}{M}$͵ηωπη

4.2 EXERCISES

Identify each numeral in Exercises 1–20 as Babylonian, Mayan, or Greek. Give the equivalent in the Hindu-Arabic system.

1.

2. 

3. 

4.

5. σλδ

6. ωοβ

7. 

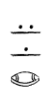
8. 

9. 

10. 

11. 

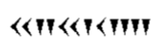
12. 

13. 

14. 

15. 

16. 

17. 

18. 

19. $\overset{\alpha}{M}$͵ερμθ

20. $\overset{\eta}{M}$ωη

Write each number as a Babylonian numeral.

21. 21 **22.** 32 **23.** 293

24. 412 **25.** 1514 **26.** 3280

27. 5190 **28.** 7842 **29.** 43,205

30. 90,180

Write each number as a Mayan numeral.

31. 12 **32.** 32 **33.** 151

34. 208 **35.** 4694 **36.** 4328

37. 64,712 **38.** 61,598

Write each number as a Greek numeral.

39. 39 **40.** 51 **41.** 92

42. 106 **43.** 412 **44.** 381

45. 2769 **46.** 9814 **47.** 54,726

48. 80,102

4.3 ARITHMETIC IN THE HINDU-ARABIC SYSTEM

Expanded Form • Historical Calculation Devices

Expanded Form

The historical development of numeration culminated in positional systems. The most successful of these is the Hindu-Arabic system, which has base ten and, therefore, has place values that are powers of 10.

We now review exponential expressions, or powers (defined in **Section 1.1**), because they are the basis of expanded form in a positional system.

EXAMPLE 1 Evaluating Powers

Find each power.

(a) 10^3 **(b)** 7^2 **(c)** 5^4

SOLUTION

(a) $10^3 = 10 \cdot 10 \cdot 10 = 1000$
(10^3 is read "10 cubed," or "10 to the third power.")

(b) $7^2 = 7 \cdot 7 = 49$
(7^2 is read "7 squared," or "7 to the second power.")

(c) $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$
(5^4 is read "5 to the fourth power.")

This Iranian stamp should remind us that counting on fingers (and toes) is an age-old practice. In fact, our word **digit,** referring to the numerals 0–9, comes from a Latin word for "finger" (or "toe"). Aristotle first noted the relationships between fingers and base ten in Greek numeration. Anthropologists go along with the notion. Some cultures, however, have used two, three, or four as number bases, for example, counting on the joints of the fingers or the spaces between them.

To simplify work with exponents, it is agreed that

$$\boldsymbol{a^0 = 1}, \quad \text{for any nonzero number } a.$$

Thus, $7^0 = 1$, $52^0 = 1$, and so on. At the same time,

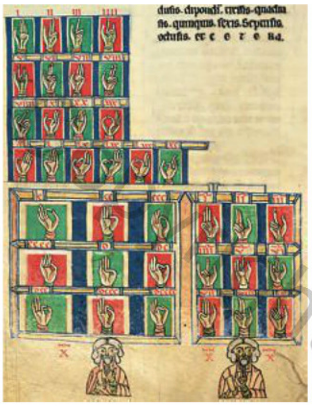
$$\boldsymbol{a^1 = a}, \quad \text{for any number } a.$$

For example, $8^1 = 8$, and $25^1 = 25$. The exponent 1 is usually omitted.

By using exponents, numbers can be written in **expanded form** in which the value of the digit in each position is made clear. For example,

$$\begin{aligned} 924 &= 900 + 20 + 4 \\ &= (9 \cdot \mathbf{100}) + (2 \cdot \mathbf{10}) + (4 \cdot \mathbf{1}) \\ &= (9 \cdot \mathbf{10^2}) + (2 \cdot \mathbf{10^1}) + (4 \cdot \mathbf{10^0}). \end{aligned}$$

$100 = 10^2$, $10 = 10^1$, and $1 = 10^0$



There is much evidence that early humans (in various cultures) used their fingers to represent numbers. As calculations became more complicated, finger reckoning, as illustrated above, became popular. The Romans became adept at this sort of calculating, carrying it to 10,000 or perhaps higher.

EXAMPLE 2 Writing Numbers in Expanded Form

Write each number in expanded form.

(a) 1906 **(b)** 46,424

SOLUTION

(a) $1906 = (1 \cdot 10^3) + (9 \cdot 10^2) + (0 \cdot 10^1) + (6 \cdot 10^0)$

Because $0 \cdot 10^1 = 0$, this term could be omitted, but the form is clearer with it included.

(b) $46{,}424 = (4 \cdot 10^4) + (6 \cdot 10^3) + (4 \cdot 10^2) + (2 \cdot 10^1) + (4 \cdot 10^0)$

EXAMPLE 3 Simplifying Expanded Numbers

Simplify each expansion.

(a) $(3 \cdot 10^5) + (2 \cdot 10^4) + (6 \cdot 10^3) + (8 \cdot 10^2) + (7 \cdot 10^1) + (9 \cdot 10^0)$

(b) $(2 \cdot 10^1) + (8 \cdot 10^0)$

SOLUTION

(a) $(3 \cdot 10^5) + (2 \cdot 10^4) + (6 \cdot 10^3) + (8 \cdot 10^2) + (7 \cdot 10^1) + (9 \cdot 10^0) = 326{,}879$

(b) $(2 \cdot 10^1) + (8 \cdot 10^0) = 28$

Expanded notation can be used to see why standard algorithms for addition and subtraction really work. The key idea behind these algorithms is based on the **distributive property.**

Distributive Property

For all real numbers *a*, *b*, and *c*,

$$(b \cdot a) + (c \cdot a) = (b + c) \cdot a.$$

For example,

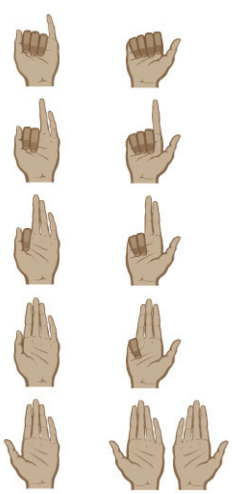
$$(3 \cdot 10^4) + (2 \cdot 10^4) = (3 + 2) \cdot 10^4 = 5 \cdot 10^4.$$

EXAMPLE 4 Adding Expanded Forms

Use expanded notation to add 23 and 64.

SOLUTION

$$\begin{aligned} 23 &= (2 \cdot 10^1) + (3 \cdot 10^0) \\ +\ 64 &= (6 \cdot 10^1) + (4 \cdot 10^0) \\ & \quad (8 \cdot 10^1) + (7 \cdot 10^0) = 87 \quad \text{Sum} \end{aligned}$$



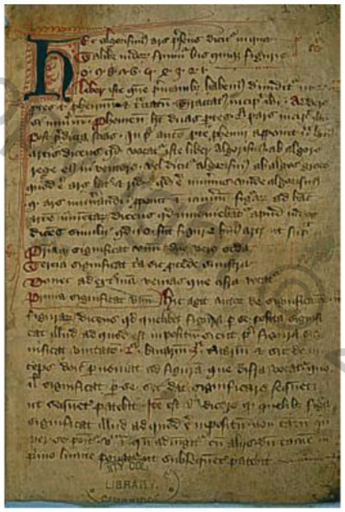
Finger Counting The first digits many people used for counting were their fingers. In Africa the Zulu used the method shown here to count to ten. They started on the left hand with palm up and fist closed. The Zulu finger positions for 1–5 are shown above on the left. The Zulu finger positions for 6–10 are shown on the right.

EXAMPLE 5 Subtracting Expanded Forms

Use expanded notation to subtract 254 from 695.

SOLUTION

$$\begin{aligned} 695 &= (6 \cdot 10^2) + (9 \cdot 10^1) + (5 \cdot 10^0) \\ -254 &= (2 \cdot 10^2) + (5 \cdot 10^1) + (4 \cdot 10^0) \\ & \quad (4 \cdot 10^2) + (4 \cdot 10^1) + (1 \cdot 10^0) = 441 \quad \text{Difference} \end{aligned}$$



The ***Carmen de Algorismo*** (opening verses shown here) by Alexander de Villa Dei, thirteenth century, popularized the new art of "algorismus":

> *. . . from these twice five figures 0 9 8 7 6 5 4 3 2 1 of the Indians we benefit . . .*

The *Carmen* related that Algor, an Indian king, invented the art. But actually, "algorism" (or "algorithm") comes in a roundabout way from the name Muhammad ibn Musa al-Khorârizmi, an Arabian mathematician of the ninth century, whose arithmetic book was translated into Latin. Furthermore, this Muhammad's book on equations, *Hisab al-jabr w'almuqâbalah,* yielded the term "algebra" in a similar way.

EXAMPLE 6 Carrying in Expanded Form

Use expanded notation to add 75 and 48.

SOLUTION

$$\begin{aligned} 75 &= (7 \cdot 10^1) + (5 \cdot 10^0) \\ +\ 48 &= (4 \cdot 10^1) + (8 \cdot 10^0) \\ \hline &\ \ (11 \cdot 10^1) + (13 \cdot 10^0) \end{aligned}$$

The units position (10^0) has room for only one digit, so we modify $13 \cdot 10^0$.

$$\begin{aligned} 13 \cdot 10^0 &= (10 \cdot 10^0) + (3 \cdot 10^0) && \text{Distributive property} \\ &= (1 \cdot 10^1) + (3 \cdot 10^0) && 10 \cdot 10^0 = 1 \cdot 10^1 \end{aligned}$$

The 1 from 13 moved to the left (carried) from the units position to the tens position.

$$\begin{aligned} &(11 \cdot 10^1) + \overbrace{(1 \cdot 10^1) + (3 \cdot 10^0)}^{13 \cdot 10^0} \\ &= (12 \cdot 10^1) + (3 \cdot 10^0) && \text{Distributive property} \\ &= (10 \cdot 10^1) + (2 \cdot 10^1) + (3 \cdot 10^0) && \text{Modify } 12 \cdot 10^1. \\ &= (1 \cdot 10^2) + (2 \cdot 10^1) + (3 \cdot 10^0) && 10 \cdot 10^1 = 1 \cdot 10^2 \\ &= 123 && \text{Sum} \end{aligned}$$

■■■

EXAMPLE 7 Borrowing in Expanded Form

Use expanded notation to subtract 186 from 364.

SOLUTION

$$\begin{aligned} 364 &= (3 \cdot 10^2) + (6 \cdot 10^1) + (4 \cdot 10^0) \\ -186 &= (1 \cdot 10^2) + (8 \cdot 10^1) + (6 \cdot 10^0) \end{aligned}$$

We cannot subtract 6 from 4. The units position borrows from the tens position.

$$\begin{aligned} &(3 \cdot 10^2) + (6 \cdot 10^1) + (4 \cdot 10^0) \\ &= (3 \cdot 10^2) + (5 \cdot 10^1) + (1 \cdot 10^1) + (4 \cdot 10^0) && \text{Distributive property} \\ &= (3 \cdot 10^2) + (5 \cdot 10^1) + (10 \cdot 10^0) + (4 \cdot 10^0) && 1 \cdot 10^1 = 10 \cdot 10^0 \\ &= (3 \cdot 10^2) + (5 \cdot 10^1) + (14 \cdot 10^0) && \text{Distributive property} \end{aligned}$$

We cannot take 8 from 5 in the tens position, so we borrow from the hundreds.

$$\begin{aligned} &(3 \cdot 10^2) + (5 \cdot 10^1) + (14 \cdot 10^0) \\ &= (2 \cdot 10^2) + (1 \cdot 10^2) + (5 \cdot 10^1) + (14 \cdot 10^0) && \text{Distributive property} \\ &= (2 \cdot 10^2) + (10 \cdot 10^1) + (5 \cdot 10^1) + (14 \cdot 10^0) && 1 \cdot 10^2 = 10 \cdot 10^1 \\ &= (2 \cdot 10^2) + (15 \cdot 10^1) + (14 \cdot 10^0) && \text{Distributive property} \end{aligned}$$

Now we can complete the subtraction.

$$\begin{aligned} &\ \ \ (2 \cdot 10^2) + (15 \cdot 10^1) + (14 \cdot 10^0) \\ &- (1 \cdot 10^2) + (8 \cdot 10^1) + (6 \cdot 10^0) \\ \hline &\ \ \ (1 \cdot 10^2) + (7 \cdot 10^1) + (8 \cdot 10^0) = 178 && \text{Difference} \end{aligned}$$

■■■

Smart phones perform mathematical calculations and many other functions as well.

Examples 4–7 used expanded notation and the distributive property to clarify our usual addition and subtraction methods. In practice, our actual work for these four problems would appear as follows.

$$\begin{array}{r} 23 \\ +\ 64 \\ \hline 87 \end{array} \qquad \begin{array}{r} 695 \\ -\ 254 \\ \hline 441 \end{array} \qquad \begin{array}{r} {\scriptstyle 1} \\ 75 \\ +\ 48 \\ \hline 123 \end{array} \qquad \begin{array}{r} {\scriptstyle 2\ 15} \\ \not{3}\,\not{6}\,{}^{1}4 \\ -\ 1\ 8\ 6 \\ \hline 1\ 7\ 8 \end{array}$$

The procedures seen in this section also work for positional systems with bases other than ten.

Historical Calculation Devices

Because our numeration system is based on powers of ten, it is often called the **decimal system,** from the Latin word *decem,* meaning ten.* Over the years, many methods have been devised for speeding calculations in the decimal system.

One of the oldest calculation methods is the **abacus,** a device made with a series of rods with sliding beads and a dividing bar. Reading from right to left, the rods have values of 1, 10, 100, 1000, and so on. The bead above the bar has five times the value of those below. Beads moved *toward* the bar are in the "active" position, and those toward the frame are ignored. In our illustrations of *abaci* (plural form of abacus), such as in **Figure 8**, the activated beads are shown in black.

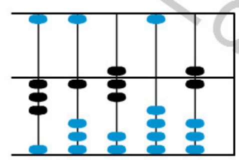


Figure 8

EXAMPLE 8 Reading an Abacus

What number is shown on the abacus in **Figure 8**?

SOLUTION

Find the number as follows.

Beads above the bar have five times the value.

$$\begin{aligned}(3 \cdot 10{,}000) &+ (1 \cdot 1000) + [(1 \cdot 500) + (2 \cdot 100)] + 0 \cdot 10 + [(1 \cdot 5) + (1 \cdot 1)] \\ &= 30{,}000 + 1000 + 500 + 200 + 0 + 5 + 1 \\ &= 31{,}706\end{aligned}$$

The **speed and accuracy of the abacus** are well known, according to www.ucmasusa.com. In a contest held between a Japanese **soroban** (the Japanese version of the abacus) expert and a highly skilled desk-calculator operator, the abacus won on addition, subtraction, division, and combinations of these operations. The electronic calculator won only on multiplication.

As paper became more readily available, people gradually switched from devices like the abacus (though these still are commonly used in some areas) to paper-and-pencil methods of calculation. One early scheme, used in India and Persia, was the **lattice method,** which arranged products of single digits into a diagonalized lattice.

**December* was the tenth month in an old form of the calendar. It is interesting to note that *decem* became *dix* in the French language; a ten-dollar bill, called a "dixie," was in use in New Orleans before the Civil War. "Dixie Land" was a nickname for that city before Dixie came to refer to all the Southern states, as in Daniel D. Emmett's song, written in 1859.

John Napier's most significant mathematical contribution, developed over a period of at least 20 years, was the concept of **logarithms,** which, among other things, allow multiplication and division to be accomplished with addition and subtraction. It was a great computational advantage given the state of mathematics at the time (1614).

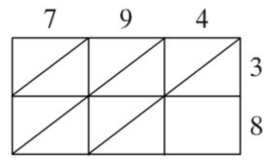
Napier, a supporter of John Knox and James I, published a widely read anti-Catholic work that analyzed the Biblical book of Revelation. He concluded that the Pope was the Antichrist and that the Creator would end the world between 1688 and 1700. Napier was one of many who, over the years, have miscalculated the end of the world.

EXAMPLE 9 Using the Lattice Method for Products

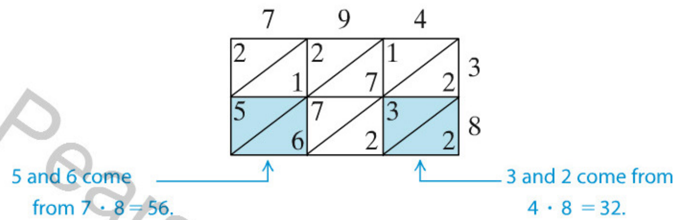
Find the product $38 \cdot 794$ by the lattice method.

SOLUTION

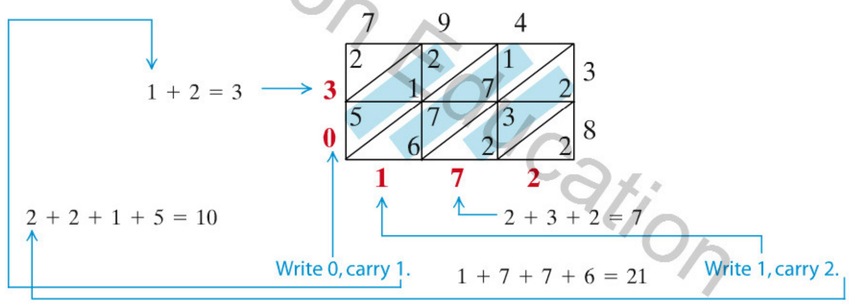
Step 1 Write the problem, with one number at the side and one across the top.



Step 2 Within the lattice, write the products of all pairs of digits from the top and side.



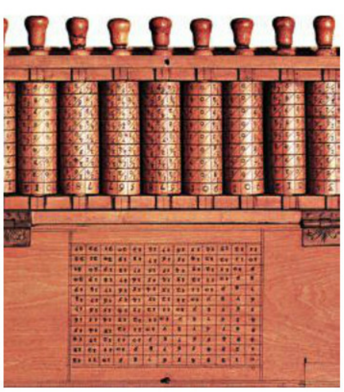
Step 3 Starting at the right of the lattice add diagonally, carrying as necessary.



Step 4 Read the answer around the left side and bottom: $38 \cdot 794 =$ **30,172.**

The Scottish mathematician John Napier (1550–1617) introduced a significant calculating tool called **Napier's rods,** or **Napier's bones.** Napier's invention, based on the lattice method of multiplication, is widely acknowledged as a very early forerunner of modern computers. It consisted of a set of strips, several for each digit 0 through 9, on which multiples of each digit appeared in a sort of lattice column. See **Figure 9**.

An additional strip, called the *index,* could be laid beside any of the others to indicate the multiplier at each level. **Figure 10** shows how to multiply 2806 by 7. Select the rods for 2, 8, 0, and 6, placing them side by side. Then using the index, locate the level for a multiplier of 7. The resulting lattice gives the product, **19,642.**



Napier's rods were an early step toward modern computers.

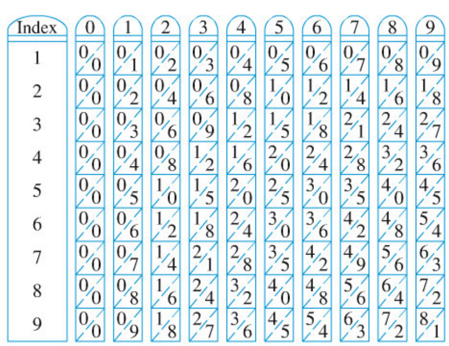


Figure 9

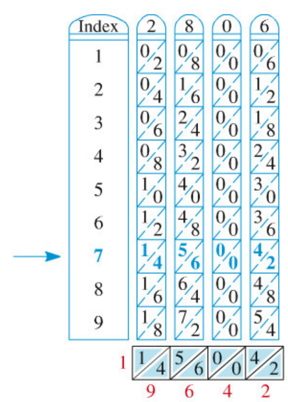


Figure 10

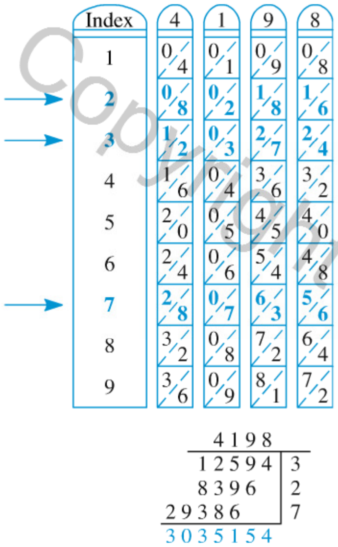


Figure 11

EXAMPLE 10 Multiplying with Napier's Rods

Use Napier's rods to find the product of 723 and 4198.

SOLUTION

We line up the rods for 4, 1, 9, and 8 next to the index, as in **Figure 11**. The product $3 \cdot 4198$ is found as described in **Example 9** and written at the bottom of the figure. Then $2 \cdot 4198$ is found similarly and written below, shifted one place to the left. (Why?) Finally, the product $7 \cdot 4198$ is written shifted two places to the left.

The final answer is found by addition.

$$723 \cdot 4198 = \mathbf{3{,}035{,}154}$$

Another paper-and-pencil method of multiplication is the **Russian peasant method,** which is similar to the Egyptian method of doubling explained in **Section 4.1.** To multiply 37 and 42 by the Russian peasant method, make two columns headed by 37 and 42. Form the first column by dividing 37 by 2 again and again, ignoring any remainders. Stop when 1 is obtained. Form the second column by doubling each number down the column.

Divide by 2, ignoring remainders. ↓			Double each number. ↓
	37	42	
	18	84	
	9	168	
	4	336	
	2	672	
	1	1344	

Now add up only the second column numbers that correspond to odd numbers in the first column. Omit those corresponding to even numbers in the first column.

→	**37**	**42**	←
	18	84	
Identify odd numbers. →	**9**	**168**	← Add these numbers.
	4	336	
	2	672	
→	**1**	**1344**	←

$$37 \cdot 42 = \mathbf{42} + \mathbf{168} + \mathbf{1344} = \mathbf{1554} \leftarrow \text{Answer}$$

Most people use standard algorithms for adding and subtracting, carrying or borrowing when appropriate, as illustrated following **Example 7.** An interesting alternative is the **nines complement method** for subtracting. To use this method, we first agree that the nines complement of a digit n is $9 - n$. For example, the nines complement of 0 is 9, of 1 is 8, of 2 is 7, and so on, up to the nines complement of 9, which is 0.

To carry out the nines complement method, complete the following steps:

Step 1 Align the digits as in the standard subtraction algorithm.

Step 2 Add leading zeros, if necessary, in the subtrahend so that both numbers have the same number of digits.

Step 3 Replace each digit in the subtrahend with its nines complement, and then add.

Step 4 Finally, delete the leading digit (1), and add 1 to the remaining part of the sum.

For a way to include a little magic with your calculations, check out http://digicc.com/fido.

EXAMPLE 11 Using the Nines Complement Method

Use the nines complement method to subtract 2803 − 647.

SOLUTION

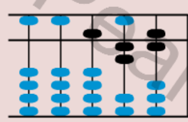
Step 1	Step 2	Step 3	Step 4	
2803	2803	2803	2155	
− 647	− 0647	+ 9352	+ 1	
		12,155	2156	Difference

For Further Thought

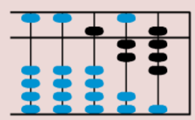
Calculating on the Abacus

The abacus has been (and still is) used to perform rapid calculations. Add 526 and 362 as shown.

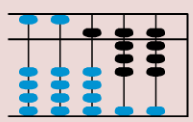
Start with 526 on the abacus.



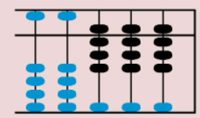
To add 362, start by "activating" an additional 2 on the 1s rod.



Next, activate an additional 6 on the 10s rod.



Finally, activate an additional 3 on the 100s rod.



The sum, read from the abacus, is 888.

For problems where carrying or borrowing is required, it takes a little more thought and skill.

For Group or Individual Investigation

1. Use an abacus to add: 13,728 + 61,455. Explain each step of your procedure.
2. Use an abacus to subtract: 6512 − 4816. Explain each step of your procedure.

4.3 EXERCISES

Write each number in expanded form.

1. 84 **2.** 352 **3.** 9446 **4.** 12,398

5. four thousand, nine hundred twenty-four

6. fifty-two thousand, one hundred eighteen

7. fourteen million, two hundred six thousand, forty

8. two hundred twelve million, eleven thousand, nine hundred sixteen

Simplify each expansion.

9. $(7 \cdot 10^1) + (5 \cdot 10^0)$

10. $(8 \cdot 10^2) + (2 \cdot 10^1) + (0 \cdot 10^0)$

11. $(4 \cdot 10^3) + (3 \cdot 10^2) + (8 \cdot 10^1) + (0 \cdot 10^0)$

12. $(5 \cdot 10^5) + (0 \cdot 10^4) + (3 \cdot 10^3) + (5 \cdot 10^2) + (6 \cdot 10^1) + (8 \cdot 10^0)$

13. $(7 \cdot 10^7) + (4 \cdot 10^5) + (1 \cdot 10^3) + (9 \cdot 10^0)$

14. $(3 \cdot 10^8) + (8 \cdot 10^6) + (2 \cdot 10^4) + (3 \cdot 10^0)$

In each of the following, add in expanded notation.

15. 37 + 42 **16.** 582 + 613

In each of the following, subtract in expanded notation.

17. 85 − 32 **18.** 724 − 423

Perform each addition using expanded notation.

19. 75 + 34 **20.** 557 + 378

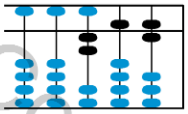
21. 434 + 299 **22.** 6755 + 4827

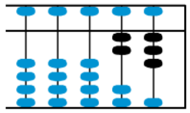
Perform each subtraction using expanded notation.

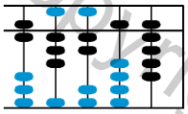
23. 54 − 48 **24.** 364 − 59

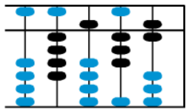
25. 645 − 439 **26.** 816 − 335

Identify the number represented on each abacus.

27. 

28. 

29. 

30. 

Sketch an abacus to show each number.

31. 38 **32.** 183 **33.** 2547 **34.** 70,163

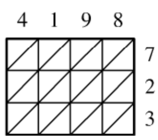
Use the lattice method to find each product.

35. $65 \cdot 29$ **36.** $32 \cdot 741$

37. $525 \cdot 73$ **38.** $912 \cdot 483$

*Refer to **Example 10** where Napier's rods were used. Then complete Exercises 39 and 40.*

39. Find the product of 723 and 4198 by completing the lattice process shown here.



40. Explain how Napier's rods could have been used in **Example 10** to set up one complete lattice product rather than adding three individual (shifted) lattice products. Illustrate with a sketch.

*Use Napier's rods (**Figure 9**) to find each product.*

41. $8 \cdot 62$ **42.** $32 \cdot 73$

43. $26 \cdot 8354$ **44.** $526 \cdot 4863$

Perform each subtraction using the nines complement method.

45. $283 - 41$ **46.** $536 - 425$

47. $50{,}000 - 199$ **48.** $40{,}002 - 4846$

Use the Russian peasant method to find each product.

49. $5 \cdot 92$ **50.** $41 \cdot 53$

51. $62 \cdot 529$ **52.** $145 \cdot 63$

The Hindu-Arabic system is positional and uses ten as the base. Describe any advantages or disadvantages that may have resulted in each case.

53. Suppose the base had been larger, say twelve or twenty.

54. Suppose the base had been smaller, maybe eight or five.

4.4 CONVERSION BETWEEN NUMBER BASES

General Base Conversions • Computer Mathematics

General Base Conversions

In this section we consider bases other than ten, but we use the familiar Hindu-Arabic symbols. We indicate bases other than ten with a spelled-out subscript, as in the numeral 43_{five}. ***Whenever a number appears without a subscript, it is assumed that the intended base is ten.*** Be careful how you read (or verbalize) numerals here. The numeral 43_{five} is read "four three base five." (Do *not* read it as "forty-three," as that terminology implies base ten and names a totally different number.)

Table 8 gives powers of some numbers used as alternative bases.

Table 8 Selected Powers of Some Alternative Number Bases

	Fourth Power	Third Power	Second Power	First Power	Zero Power
Base two	16	8	4	2	1
Base five	625	125	25	5	1
Base seven	2401	343	49	7	1
Base eight	4096	512	64	8	1
Base sixteen	65,536	4096	256	16	1