14.2
Limits and Continuity

In this section, we will learn about:

Limits and continuity of

various types of functions.

LIMITS AND CONTINUITY

- Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as *x* and *y* both approach 0 (and thus the point (*x*, *y*) approaches the origin).

LIMITS AND CONTINUITY

- The following tables show values of *f*(*x*, *y*) and *g*(*x*, *y*), correct to three decimal places, for points (*x*, *y*) near the origin.

LIMITS AND CONTINUITY Table 1

•This table shows values of *f*(*x*, *y*).

$f = \frac{\sin(x^2+y^2)}{x^2+y^2}$

TABLE 1 Values of $f(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

y=0, y=−x, y=x, x=0

LIMITS AND CONTINUITY Table 2

•This table shows values of *g*(*x*, *y*).

$g = \frac{x^2-y^2}{x^2+y^2}$

along y=0 $g \to 1$

along x=0 $g = \frac{x^2-y^2}{x^2+y^2} \to -1$

DNE

TABLE 2 Values of $g(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

y=0, x=0

LIMITS AND CONTINUITY

- Notice that neither function is defined at the origin.
  - It appears that, as (*x*, *y*) approaches (0, 0), the values of *f*(*x*, *y*) are approaching 1 whereas the values of *g*(*x*, *y*) aren't approaching any number.

LIMITS AND CONTINUITY

- It turns out that these guesses based on numerical evidence are correct.
- Thus, we write:
  - $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = 1$
  - $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

$\lim_{x\to 0}\frac{\sin x}{x}=1$ $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$ $\lim_{r\to 0}\frac{\sin(r^2)}{r^2} = \boxed{1}$

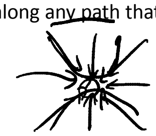
LIMITS AND CONTINUITY

- In general, we use the notation

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

to indicate that:

- The values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f.



LIMITS AND CONTINUITY

- In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). ε-δ

LIMIT OF A FUNCTION

Definition 1

- Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b).
- Then, we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L.

SINGLE VARIABLE FUNCTIONS

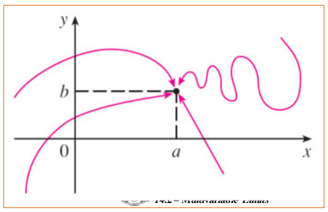
- For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right.
  - We recall from Chapter 2 that, if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ does not exist.

DOUBLE VARIABLE FUNCTIONS

- For functions of two variables, the situation is not as simple.

DOUBLE VARIABLE FUNCTIONS

- This is because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever as long as (x, y) stays within the domain of f.



LIMIT OF A FUNCTION

- Definition 1 refers only to the distance between (x, y) and (a, b).
  - It does not refer to the direction of approach.

LIMIT OF A FUNCTION

- Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b).

LIMIT OF A FUNCTION

- Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

LIMIT OF A FUNCTION

- If
  $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and
  $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2,
  where $L_1 \neq L_2$,
  then

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

does not exist.

LIMIT OF A FUNCTION

Example 1

- Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{DNE}$$

does not exist.

- Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.

Along $x=0$: $g \to \frac{-y^2}{y^2} = -1$, $\lim_{y\to 0} = -1$

Along $y=0$: $g \to \frac{x^2}{x^2} = 1$, $\lim_{x\to 0} = 1$

$-1 \neq 1$

LIMIT OF A FUNCTION

Example 1

- First, let's approach (0, 0) along the x-axis.
  - Then, $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$.
  - So, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis.

LIMIT OF A FUNCTION

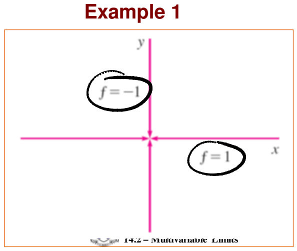
Example 1

- We now approach along the y-axis by putting $x = 0$.
  - Then, $f(0, y) = -y^2/y^2 = -1$ for all $y \neq 0$.
  - So, $f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis.

LIMIT OF A FUNCTION

- Since f has two different limits along two different lines, the given limit does not exist.
  - This confirms the conjecture we made on the basis of numerical evidence at the beginning of the section.

Example 1



LIMIT OF A FUNCTION

Example 2

$g = \frac{x^2-y^2}{x^2+y^2}$

- If

$$h(x, y) = \frac{xy}{x^2 + y^2}$$

① along $x=0$ $h \to 0$

② along $y=0$ $h \to 0$

does

$$\lim_{(x,y)\to(0,0)} h(x, y)$$

DNE

exist?

③ along $y=x$

$h \to \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$

LIMIT OF A FUNCTION

Example 2

- If $y = 0$, then $f(x, 0) = 0/x^2 = 0$.
  - Therefore,

    $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the x-axis.

LIMIT OF A FUNCTION

Example 2

- If $x = 0$, then $f(0, y) = 0/y^2 = 0$.
  - So,

    $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the y-axis.

LIMIT OF A FUNCTION

Example 2

- Although we have obtained identical limits along the axes, that does not show that the given limit is 0.

LIMIT OF A FUNCTION

Example 2

- Let's now approach (0, 0) along another line, say $y = x$.
  - For all $x \neq 0$,

$$f(x,x)=\frac{x^2}{x^2+x^2}=\frac{1}{2}$$

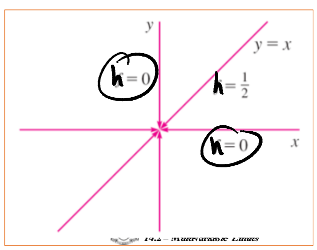
  - Therefore,

$$f(x,y)\to\tfrac{1}{2} \text{ as } (x,y)\to(0,0) \text{ along } y=x$$

LIMIT OF A FUNCTION

Example 2

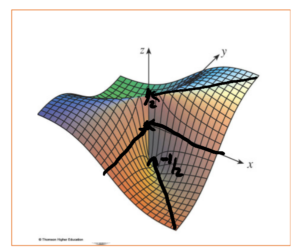
- Since we have obtained different limits along different paths, the given limit does not exist.



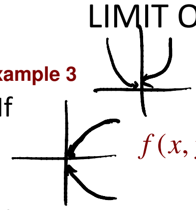
LIMIT OF A FUNCTION

•This figure sheds some light on Example 2.

- The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = ½$ for all points (x, y) on that line except the origin.



LIMIT OF A FUNCTION



Example 3

- If

$$f(x,y)=\frac{xy^2}{x^2+y^4}$$

$x(x^2)^2$ / $(x^2)^4$

does

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

DNE

exist?

Along	f
① $x=0$	$f\to 0$
② $y=0$	$f\to 0$
③ $y=x$	$f\to\frac{x^3}{x^2+x^4}\to 0$ as $x\to 0$
④ $y=-x$	$f\to\frac{x^3}{x^2+x^4}\to 0$ as $x\to 0$
⑤ $y=x^2$	$f\to\frac{x^5}{x^2+x^8}\to 0$ as $x\to 0$
⑥ $x=y^2$	$f\to\frac{y^4}{2y^4}\to\frac{1}{2}$

$$\frac{y^2\cdot y^2}{(y^2)^2+y^4}=\frac{y^4}{2y^4}$$

y^4

LIMIT OF A FUNCTION

Example 3

- With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \to (0, 0)$ along any nonvertical line through the origin.

LIMIT OF A FUNCTION

Example 3

- Then, $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

LIMIT OF A FUNCTION

Example 3

- Therefore,

  $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along $y = mx$

  – Thus, f has the same limiting value along every nonvertical line through the origin.

LIMIT OF A FUNCTION

Example 3

- However, that does not show that the given limit is 0.

  – This is because, if we now let $(x, y) \to (0, 0)$ along the parabola $x = y^2$ we have:

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

  – So, $f(x, y) \to ½$ as $(x, y) \to (0, 0)$ along $x = y^2$

LIMIT OF A FUNCTION

Example 3

- Since different paths lead to different limiting values, the given limit does not exist.

LIMIT OF A FUNCTION

- Now, let's look at limits that do exist.

LIMIT OF A FUNCTION

- Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits.

LIMIT OF A FUNCTION

- The Limit Laws listed in Section 2.3 can be extended to functions of two variables.
- For instance,
  - The limit of a sum is the sum of the limits.
  - The limit of a product is the product of the limits.

LIMIT OF A FUNCTION

Equations 2

- In particular, the following equations are true.

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

$$\lim_{(x,y)\to(a,b)} c = c$$

LIMIT OF A FUNCTION

Equations 2

- The Squeeze Theorem also holds.

CONTINUITY OF SINGLE VARIABLE FUNCTIONS

- Recall that evaluating limits of continuous functions of a single variable is easy.
  - It can be accomplished by direct substitution.
  - This is because the defining property of a continuous function is

$$\lim_{x\to a} f(x) = f(a)$$

11. $\displaystyle \lim_{(x,y)\to(1,\pi/6)} \frac{x \sin y}{x^2+1} = \frac{1\cdot\sin(\pi/6)}{1+1} = \frac{1/2}{2} = \boxed{\frac{1}{4}}$

$x=1$
$y=\pi/6$

① Plug in $x=a$, $y=b$ to the denominator

16. $\displaystyle \lim_{\substack{(x,y)\to(2,-4)\\ y\neq-4,\ x\neq x^2}} \frac{y+4}{x^2y-xy+4x^2-4x}$

$2^2(-4)-2(-4)+4(2)^2-4(2)$
$\cancel{4(-4)}+\cancel{8}+\cancel{16}-\cancel{8}=0$

$y+4\to 0$

Factor by Grouping

$x^2y-xy+4x^2-4x$
$y(x^2-x)+4(x^2-x)$
$(y+4)(x^2-x)$

$\displaystyle \lim_{(x,y)\to(2,-4)} \frac{\cancel{y+4}}{\cancel{(y+4)}(x^2-x)} = \lim_{(x,y)\to(2,-4)} \frac{1}{x^2-x} = \frac{1}{2}$

2^2-2

23. $\displaystyle \lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$

Sum of cubes

$x=1$
$y=-1$
den $\to 0$
num $\to 0$

Factor

$\boxed{x^3+y^3=(x+y)(x^2-xy+y^2)}$

$\displaystyle = \lim_{(x,y)\to(1,-1)} \frac{\cancel{(x+y)}(x^2-xy+y^2)}{\cancel{(x+y)}}$

$\displaystyle = \lim_{(x,y)\to(1,-1)} x^2-xy+y^2$

$1+1+1$

$=3$

num $= \sqrt{4} - \sqrt{3+1} = 0$

20. $\displaystyle\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$

4 − 3

den $= 1 - 1 = 0$

Mult. by the Conjugate

$$\lim_{(x,y)\to(4,3)} \frac{(\sqrt{x}-\sqrt{y+1})}{(x-y-1)}\frac{(\sqrt{x}+\sqrt{y+1})}{(\sqrt{x}+\sqrt{y+1})} = \lim_{(x,y)\to(4,3)} \frac{\cancel{x-y-1}\; x-(y+1)}{\cancel{(x-y-1)}(\sqrt{x}+\sqrt{y+1})}$$

$$\lim_{(x,y)\to(4,3)} \frac{1}{\sqrt{x}+\sqrt{y+1}} = \frac{1}{4}$$

$\sqrt{4}+\sqrt{3+1}$

$2+2$

CONTINUITY OF DOUBLE VARIABLE FUNCTIONS

- Continuous functions of two variables are also defined by the direct substitution property.

Math 114 – Rimmer
14.2 – Multivariable Limits

CONTINUITY

Definition 4

- A function *f* of two variables is called continuous at (*a*, *b*) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

- We say *f* is continuous on *D* if *f* is continuous at every point (*a*, *b*) in *D*.

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14.2 – Multivariable Limits

CONTINUITY

- The intuitive meaning of continuity is that, if the point (*x*, *y*) changes by a small amount, then the value of *f*(*x*, *y*) changes by a small amount.
  - This means that a surface that is the graph of a continuous function has no hole or break.

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14.2 – Multivariable Limits

CONTINUITY

- Using the properties of limits, you can see that sums, differences, products, quotients of continuous functions are continuous on their domains.
  - Let's use this fact to give examples of continuous functions.

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14.2 – Multivariable Limits

POLYNOMIAL

- A polynomial function of two variables (polynomial, for short) is a sum of terms of the form $cx^m y^n$, where:
  - *c* is a constant.
  - *m* and *n* are nonnegative integers.

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14.2 – Multivariable Limits

RATIONAL FUNCTION

- A rational function is a ratio of polynomials.

RATIONAL FUNCTION VS. POLYNOMIAL

$f(x,y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$

- is a polynomial.

$g(x,y) = \dfrac{2xy+1}{x^2+y^2}$

- is a rational function.

CONTINUITY

- The limits in Equations 2 show that the functions

  $f(x, y) = x$, $g(x, y) = y$, $h(x, y) = c$

  are continuous.

CONTINUOUS POLYNOMIALS

- Any polynomial can be built up out of the simple functions *f*, *g*, and *h* by multiplication and addition.
  - It follows that all polynomials are continuous on $\mathbb{R}^2$.

CONTINUOUS RATIONAL FUNCTIONS

- Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

CONTINUITY

Example 5

- Evaluate

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$$

  - $f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial.
  - Thus, it is continuous everywhere.

CONTINUITY

Example 5

– Hence, we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$$
$$= 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3\cdot 1 + 2\cdot 2$$
$$= 11$$

CONTINUITY

Example 6

- Where is the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

continuous?

CONTINUITY

Example 6

- The function *f* is discontinuous at (0, 0) because it is not defined there.
- Since *f* is a rational function, it is continuous on its domain, which is the set
  $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$

CONTINUITY

Example 7

- Let

$$g(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

– Here, *g* is defined at (0, 0).

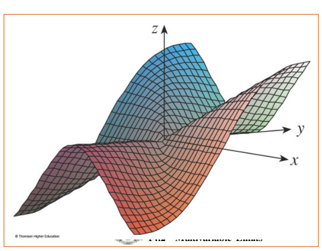
– However, it is still discontinuous there because

$$\lim_{(x,y)\to(0,0)} g(x,y)$$

does not exist (see Example 1).

CONTINUITY

- This figure shows the graph of the continuous function in Example 8.



COMPOSITE FUNCTIONS

- Just as for functions of one variable, composition is another way of combining two continuous functions to get a third.

COMPOSITE FUNCTIONS

- In fact, it can be shown that, if f is a continuous function of two variables and g is a continuous function of a single variable defined on the range of f, then
  - The composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

COMPOSITE FUNCTIONS

Example 9

- Where is the function $h(x, y) = \arctan(y/x)$ continuous?
  - The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$.
  - The function $g(t) = \arctan t$ is continuous everywhere.

COMPOSITE FUNCTIONS

Example 9

–So, the composite function

$$g(f(x, y)) = \arctan(y/ x) = h(x, y)$$

is continuous except where $x = 0$.

COMPOSITE FUNCTIONS **Example 9**

•The figure shows the break in the graph of h above the y-axis.