10

DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATIONS

We have looked at first-order differential equations from a geometric point of view (direction fields) and from a numerical point of view (Euler's method).

- What about the symbolic point of view?

DIFFERENTIAL EQUATIONS

It would be nice to have an explicit formula for a solution of a differential equation.

- Unfortunately, that is not always possible.

DIFFERENTIAL EQUATIONS

10.3
Separable Equations

In this section, we will learn about:
Certain differential equations
that can be solved explicitly.

SEPARABLE EQUATION

A separable equation is a first-order differential equation in which the expression for *dy*/*dx* can be factored as a function of *x* times a function of *y*.

- In other words, it can be written in the form $\frac{dy}{dx} = g(x)f(y)$

SEPARABLE EQUATIONS

The name separable comes from the fact that the expression on the right side can be "separated" into a function of *x* and a function of *y*.

SEPARABLE EQUATIONS   **Equation 1**

Equivalently, if *f*(*y*) ≠ 0, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$

SEPARABLE EQUATIONS

To solve this equation, we rewrite it in the differential form

$$h(y)\ dy = g(x)\ dx$$

so that:

- All *y*'s are on one side of the equation.
- All *x*'s are on the other side.

SEPARABLE EQUATIONS   **Equation 2**

Then, we integrate both sides of the equation:

$$\int h(y)\,dy = \int g(x)\,dx$$

SEPARABLE EQUATIONS

Equation 2 defines *y* implicitly as a function of *x*.

- In some cases, we may be able to solve for *y* in terms of *x*.

SEPARABLE EQUATIONS

We use the Chain Rule to justify this procedure.

- If *h* and *g* satisfy Equation 2, then

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

SEPARABLE EQUATIONS

- Thus,

$$\frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$$

- This gives:

$$h(y)\frac{dy}{dx} = g(x)$$

- Thus, Equation 1 is satisfied.

SEPARABLE EQUATIONS **Example 1**

a. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

b. Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

SEPARABLE EQUATIONS **Example 1 a**

We write the equation in terms of differentials and integrate both sides:

$$y^2\,dy = x^2\,dx$$

$$\int y^2\,dy = \int x^2\,dx$$

$$\tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C$$

where C is an arbitrary constant.

SEPARABLE EQUATIONS **Example 1 a**

We could have used a constant C_1 on the left side and another constant C_2 on the right side.

- However, then, we could combine these constants by writing $C = C_2 - C_1$.

SEPARABLE EQUATIONS **Example 1 a**

Solving for y, we get:

$$y = \sqrt[3]{x^3 + 3C}$$

- We could leave the solution like this or we could write it in the form
$$y = \sqrt[3]{x^3 + K}$$
where $K = 3C$.
- Since C is an arbitrary constant, so is K.

SEPARABLE EQUATIONS **Example 1 b**

If we put $x = 0$ in the general solution in (a), we get:

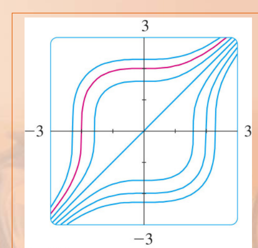
$$y(0) = \sqrt[3]{K}$$

- To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$, and so $K = 8$.
- So, the solution of the initial-value problem is:
$$y = \sqrt[3]{x^3 + 8}$$

SEPARABLE EQUATIONS

The figure shows graphs of several members of the family of solutions of the differential equation in Example 1.

- The solution of the initial-value problem in (b) is shown in red.



SEPARABLE EQUATIONS Example 2

Solve the differential equation

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$$

SEPARABLE EQUATIONS E. g. 2—Equation 3

Writing the equation in differential form and integrating both sides, we have:

$$(2y + \cos y)\, dy = 6x^2\, dx$$
$$\int (2y + \cos y)\, dy = \int 6x^2\, dx$$
$$y^2 + \sin y = 2x^3 + C$$

where C is a constant.

SEPARABLE EQUATIONS Example 2

Equation 3 gives the general solution implicitly.

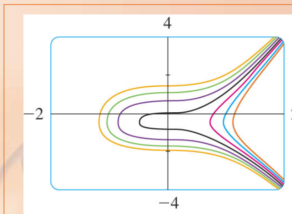
- In this case, it's impossible to solve the equation to express y explicitly as a function of x.

SEPARABLE EQUATIONS

The figure shows the graphs of several members of the family of solutions of the differential equation in Example 2.

- As we look at the curves from left to right, the values of C are:

  3, 2, 1, 0, -1, -2, -3



SEPARABLE EQUATIONS Example 3

Solve the equation

$$y' = x^2 y$$

- First, we rewrite the equation using Leibniz notation: $\frac{dy}{dx} = x^2 y$

SEPARABLE EQUATIONS Example 3

If $y \neq 0$, we can rewrite it in differential notation and integrate:

$$\frac{dy}{y} = x^2 dx \qquad y \neq 0$$
$$\int \frac{dy}{y} = \int x^2 dx$$
$$\ln|y| = \frac{x^3}{3} + C$$

SEPARABLE EQUATIONS **Example 3**

The equation defines *y* implicitly as a function of *x*.

However, in this case, we can solve explicitly for *y*.

$$|y| = e^{\ln|y|} = e^{(x^3/3)+C} = e^C e^{x^3/3}$$

Hence, $y = \pm e^C e^{x^3/3}$

SEPARABLE EQUATIONS **Example 3**

We can easily verify that the function $y = 0$ is also a solution of the given differential equation.

- So, we can write the general solution in the form

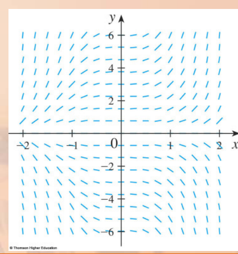
$$y = Ae^{x^3/3}$$

where A is an arbitrary constant ($A = e^C$, or $A = -e^C$, or $A = 0$).

SEPARABLE EQUATIONS

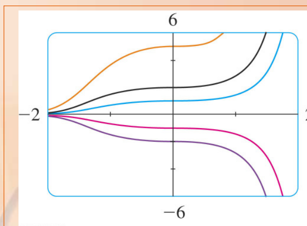
The figure shows a direction field for the differential equation in Example 3.

- Compare it with the next figure, in which we use the equation $y = Ae^{x^3/3}$ to graph solutions for several values of *A*.



SEPARABLE EQUATIONS

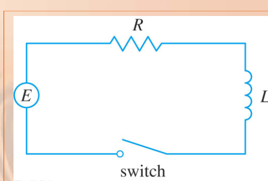
If you use the direction field to sketch solution curves with *y*-intercepts 5, 2, 1, –1, and –2, they will resemble the curves in the figure.



SEPARABLE EQUATIONS **Example 4**

In Section 9.2, we modeled the current $I(t)$ in this electric circuit by the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

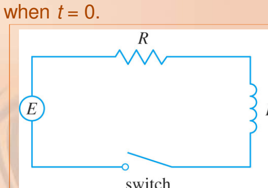


SEPARABLE EQUATIONS **Example 4**

Find an expression for the current in a circuit where:

- The resistance is 12 Ω.
- The inductance is 4 H.
- A battery gives a constant voltage of 60 V.
- The switch is turned on when $t = 0$.

What is the limiting value of the current?



SEPARABLE EQUATIONS Example 4

With L = 4, R = 12 and $E(t)$ = 60,

- The equation becomes:

$$4\frac{dI}{dt}+12I=60 \quad \text{or} \quad \frac{dI}{dt}=15-3I$$

- The initial-value problem is:

$$\frac{dI}{dt}=15-3I \qquad I(0)=0$$

SEPARABLE EQUATIONS Example 4

We recognize this as being separable.

We solve it as follows:

$$\int \frac{dI}{15-3I}=\int dt \qquad (15-3I\neq 0)$$

$$-\tfrac{1}{3}\ln|15-3I|=t+C$$

$$|15-3I|=e^{-3(t+C)}$$

$$15-3I=\pm e^{-3C}e^{-3t}=Ae^{-3t}$$

$$I=5-\tfrac{1}{3}Ae^{-3t}$$

SEPARABLE EQUATIONS Example 4

Since $I(0)$ = 0, we have:

$$5-\tfrac{1}{3}A=0$$

So, A = 15 and the solution is:

$$I(t)=5-5e^{-3t}$$

SEPARABLE EQUATIONS Example 4

The limiting current, in amperes, is:

$$\lim_{t\to\infty} I(t)=\lim_{t\to\infty}\left(5-5e^{-3t}\right)$$

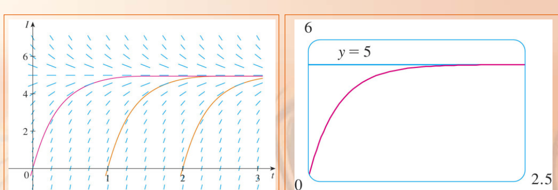
$$=5-5\lim_{t\to\infty}e^{-3t}$$

$$=5-0$$

$$=5$$

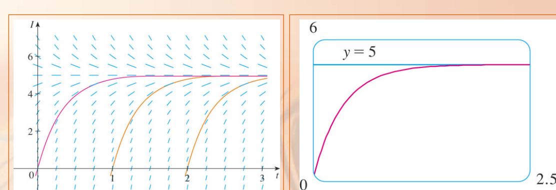
SEPARABLE EQUATIONS

The figure shows how the solution in Example 4 (the current) approaches its limiting value.



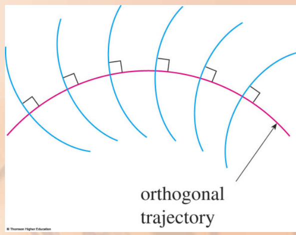
SEPARABLE EQUATIONS

Comparison with the other figure (from Section 9.2) shows that we were able to draw a fairly accurate solution curve from the direction field.



ORTHOGONAL TRAJECTORY

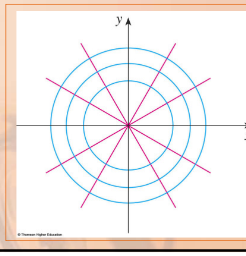
An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally—that is, at right angles.



ORTHOGONAL TRAJECTORIES

Each member of the family $y = mx$ of straight lines through the origin is an orthogonal trajectory of the family $x^2 + y^2 = r^2$ of concentric circles with center the origin.

- We say that the two families are orthogonal trajectories of each other.



ORTHOGONAL TRAJECTORIES **Example 5**

Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

ORTHOGONAL TRAJECTORIES **Example 5**

The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis.

- The first step is to find a single differential equation that is satisfied by all members of the family.

ORTHOGONAL TRAJECTORIES **Example 5**

If we differentiate $x = ky^2$, we get:

$$1 = 2ky\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{2ky}$$

- This differential equation depends on k.
- However, we need an equation that is valid for all values of k simultaneously.

ORTHOGONAL TRAJECTORIES **Example 5**

To eliminate k, we note that:

- From the equation of the given general parabola $x = ky^2$, we have $k = x/y^2$.

ORTHOGONAL TRAJECTORIES **Example 5**

Hence, the differential equation can be written as: $\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\frac{x}{y^2}y}$

or $\frac{dy}{dx} = \frac{y}{2x}$

- This means that the slope of the tangent line at any point (x, y) on one of the parabolas is: $y' = y/(2x)$

ORTHOGONAL TRAJECTORIES

On an orthogonal trajectory, the slope of the tangent line must be the negative reciprocal of this slope.

- So, the orthogonal trajectories must satisfy the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$

ORTHOGONAL TRAJECTORIES **E. g. 5—Equation 4**

The differential equation is separable.

We solve it as follows:

$$\int y\,dy = -\int 2x\,dx$$

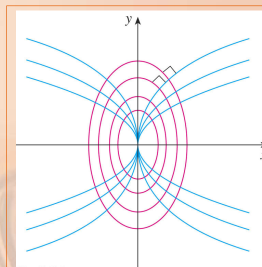
$$\frac{y^2}{2} = -x^2 + C$$

$$x^2 + \frac{y^2}{2} = C$$

where C is an arbitrary positive constant.

ORTHOGONAL TRAJECTORIES **Example 5**

Thus, the orthogonal trajectories are the family of ellipses given by Equation 4 and sketched here.



ORTHOGONAL TRAJECTORIES IN PHYSICS

Orthogonal trajectories occur in various branches of physics.

- In an electrostatic field, the lines of force are orthogonal to the lines of constant potential.
- The streamlines in aerodynamics are orthogonal trajectories of the velocity-equipotential curves.

MIXING PROBLEMS

A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance, such as salt.

- A solution of a given concentration enters the tank at a fixed rate.
- The mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate.

MIXING PROBLEMS

If $y(t)$ denotes the amount of substance in the tank at time t, then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed.

- The mathematical description of this situation often leads to a first-order separable differential equation.

MIXING PROBLEMS

We can use the same type of reasoning to model a variety of phenomena:

- Chemical reactions
- Discharge of pollutants into a lake
- Injection of a drug into the bloodstream

MIXING PROBLEMS **Example 6**

A tank contains 20 kg of salt dissolved in 5000 L of water.

- Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min.
- The solution is kept thoroughly mixed and drains from the tank at the same rate.
- How much salt remains in the tank after half an hour?

MIXING PROBLEMS **Example 6**

Let $y(t)$ be the amount of salt (in kilograms) after t minutes.

We are given that $y(0) = 20$ and we want to find $y(30)$.

- We do this by finding a differential equation satisfied by $y(t)$.

MIXING PROBLEMS **Equation 5**

Note that dy/dt is the rate of change of the amount of salt.

Thus,

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

where:

- 'Rate in' is the rate at which salt enters the tank.
- 'Rate out' is the rate at which it leaves the tank.

RATE IN **Example 6**

We have:

$$\text{rate in} = \left(0.03\frac{\text{kg}}{\text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right)$$

$$= 0.75\frac{\text{kg}}{\text{min}}$$

MIXING PROBLEMS **Example 6**

The tank always contains 5000 L of liquid.

- So, the concentration at time t is $y(t)/5000$ (measured in kg/L).

RATE OUT **Example 6**

As the brine flows out at a rate of 25 L/min, we have:

$$\text{rate out} = \left(\frac{y(t)}{5000}\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\,\frac{\text{kg}}{\text{min}}$$

MIXING PROBLEMS **Example 6**

Thus, from Equation 5, we get:

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

- Solving this separable differential equation, we obtain:

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - y| = \frac{t}{200} + C$$

MIXING PROBLEMS

Since $y(0) = 20$, we have:

$-\ln 130 = C$

So,

$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

MIXING PROBLEMS **Example 6**

Therefore,

$$|150 - y| = 130e^{-t/200}$$

- $y(t)$ is continuous and $y(0) = 20$, and the right side is never 0.
- We deduce that $150 - y(t)$ is always positive.

MIXING PROBLEMS **Example 6**

Thus, $|150 - y| = 150 - y$.

So,

$$y(t) = 150 - 130e^{-t/200}$$

- The amount of salt after 30 min is:

$$y(30) = 150 - 130e^{-30/200} \approx 38.1 \text{ kg}$$

MIXING PROBLEMS **Example 6**

Here's the graph of the function $y(t)$ of Example 6.

- Notice that, as time goes by, the amount of salt approaches 150 kg.