

DIFFERENTIAL EQUATIONS

Perhaps the most important of all the applications of calculus is to differential equations.

DIFFERENTIAL EQUATIONS

When physical or social scientists use calculus, more often than not, it is to analyze a differential equation that has arisen in the process of modeling some phenomenon they are studying.

DIFFERENTIAL EQUATIONS

It is often impossible to find an explicit formula for the solution of a differential equation.

- Nevertheless, we will see that graphical and numerical approaches provide the needed information.

DIFFERENTIAL EQUATIONS

10.1 Modeling with Differential Equations

In this section, we will learn:
How to represent some mathematical models
in the form of differential equations.

MODELING WITH DIFFERENTIAL EQUATIONS

In describing the process of modeling in Section 1.2, we talked about formulating a mathematical model of a real-world problem through either:

- Intuitive reasoning about the phenomenon
- A physical law based on evidence from experiments

DIFFERENTIAL EQUATION

The model often takes the form of a differential equation.

- This is an equation that contains an unknown function and some of its derivatives.

MODELING WITH DIFFERENTIAL EQUATIONS

This is not surprising.

- In a real-world problem, we often notice that changes occur, and we want to predict future behavior on the basis of how current values change.

MODELING WITH DIFFERENTIAL EQUATIONS

Let's begin by examining several examples of how differential equations arise when we model physical phenomena.

MODELS OF POPULATION GROWTH

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population.

MODELS OF POPULATION GROWTH

That is a reasonable assumption for a population of bacteria or animals under ideal conditions, such as:

- Unlimited environment
- Adequate nutrition
- Absence of predators
- Immunity from disease

MODELS OF POPULATION GROWTH

Let's identify and name the variables in this model:

- t = time (independent variable)
- P = the number of individuals in the population (dependent variable)

MODELS OF POPULATION GROWTH

The rate of growth of the population is the derivative dP/dt .

POPULATION GROWTH MODELS Equation 1

Hence, our assumption that the rate of growth of the population is proportional to the population size is written as the equation

$$\frac{dP}{dt} = kP$$

where k is the proportionality constant.

POPULATION GROWTH MODELS

Equation 1 is our first model for population growth.

- It is a differential equation because it contains an unknown function P and its derivative dP/dt .

POPULATION GROWTH MODELS

Having formulated a model, let's look at its consequences.

POPULATION GROWTH MODELS

If we rule out a population of 0, then

$$P(t) > 0 \text{ for all } t$$

So, if $k > 0$, then Equation 1 shows that:

$$P'(t) > 0 \text{ for all } t$$

POPULATION GROWTH MODELS

This means that the population is always increasing.

- In fact, as $P(t)$ increases, Equation 1 shows that dP/dt becomes larger.
- In other words, the growth rate increases as the population increases.

POPULATION GROWTH MODELS

Equation 1 asks us to find a function whose derivative is a constant multiple of itself.

- We know from Chapter 3 that exponential functions have that property.
- In fact, if we let $P(t) = Ce^{kt}$, then

$$P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$$

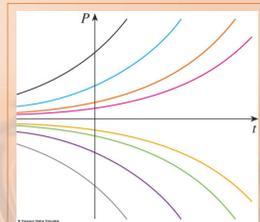
POPULATION GROWTH MODELS

Thus, any exponential function of the form $P(t) = Ce^{kt}$ is a solution of Equation 1.

- In Section 9.4, we will see that there is no other solution.

POPULATION GROWTH MODELS

Allowing C to vary through all the real numbers, we get the family of solutions $P(t) = Ce^{kt}$, whose graphs are shown.



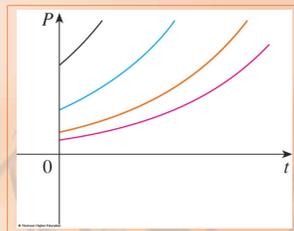
POPULATION GROWTH MODELS

However, populations have only positive values.

- So, we are interested only in the solutions with $C > 0$.
- Also, we are probably concerned only with values of t greater than the initial time $t = 0$.

POPULATION GROWTH MODELS

The figure shows the physically meaningful solutions.



POPULATION GROWTH MODELS

Putting $t = 0$, we get:

$$P(0) = Ce^{k(0)} = C$$

- The constant C turns out to be the initial population, $P(0)$.

POPULATION GROWTH MODELS

Equation 1 is appropriate for modeling population growth under ideal conditions.

However, we have to recognize that a more realistic model must reflect the fact that a given environment has limited resources.

POPULATION GROWTH MODELS

Many populations start by increasing in an exponential manner.

However, the population levels off when it approaches its carrying capacity K (or decreases toward K if it ever exceeds K .)

POPULATION GROWTH MODELS

For a model to take into account both trends, we make two assumptions:

1. $\frac{dP}{dt} \approx kP$ if P is small.
(Initially, the growth rate is proportional to P .)
2. $\frac{dP}{dt} < 0$ if $P > K$.
(P decreases if it ever exceeds K .)

POPULATION GROWTH MODELS Equation 2

A simple expression that incorporates both assumptions is given by the equation

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K} \right)$$

- If P is small compared with K , then P/K is close to 0. So, $dP/dt \approx kP$
- If $P > K$, then $1 - P/K$ is negative. So, $dP/dt < 0$

LOGISTIC DIFFERENTIAL EQUATION

Equation 2 is called the logistic differential equation.

- It was proposed by the Dutch mathematical biologist Pierre-François Verhulst in the 1840s—as a model for world population growth.

LOGISTIC DIFFERENTIAL EQUATIONS

In Section 9.4, we will develop techniques that enable us to find explicit solutions of the logistic equation.

- For now, we can deduce qualitative characteristics of the solutions directly from Equation 2.

POPULATION GROWTH MODELS

We first observe that the constant functions $P(t) = 0$ and $P(t) = K$ are solutions.

- This is because, in either case, one of the factors on the right side of Equation 2 is zero.

EQUILIBRIUM SOLUTIONS

This certainly makes physical sense. If the population is ever either 0 or at the carrying capacity, it stays that way.

- These two constant solutions are called equilibrium solutions.

POPULATION GROWTH MODELS

If the initial population $P(0)$ lies between 0 and K , then the right side of Equation 2 is positive.

- So, $dP/dt > 0$ and the population increases.

POPULATION GROWTH MODELS

However, if the population exceeds the carrying capacity ($P > K$), then $1 - P/K$ is negative.

- So, $dP/dt < 0$ and the population decreases.

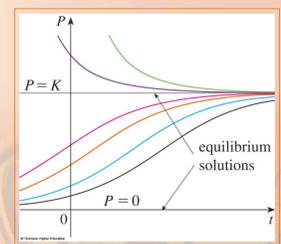
POPULATION GROWTH MODELS

Notice that, in either case, if the population approaches the carrying capacity ($P \rightarrow K$), then $dP/dt \rightarrow 0$.

- This means the population levels off.

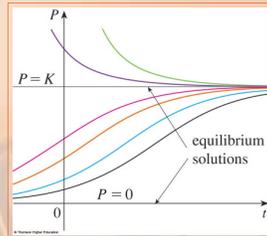
POPULATION GROWTH MODELS

So, we expect that the solutions of the logistic differential equation have graphs that look something like these.



POPULATION GROWTH MODELS

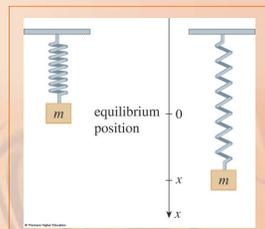
Notice that the graphs move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = K$.

**MODELING WITH DIFFERENTIAL EQUATIONS**

Let's now look at an example of a model from the physical sciences.

MODEL FOR MOTION OF A SPRING

We consider the motion of an object with mass m at the end of a vertical spring.

**MODEL FOR MOTION OF A SPRING**

In Section 6.4, we discussed Hooke's Law.

- If the spring is stretched (or compressed) x units from its natural length, it exerts a force proportional to x :

$$\text{restoring force} = -kx$$

where k is a positive constant (the spring constant).

SPRING MOTION MODEL Equation 3

If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law, we have:

$$m \frac{d^2 x}{dt^2} = -kx$$

SECOND-ORDER DIFFERENTIAL EQUATION

This is an example of a second-order differential equation.

- It involves second derivatives.

SPRING MOTION MODEL

Let's see what we can guess about the form of the solution directly from the equation.

SPRING MOTION MODEL

We can rewrite Equation 3 in the form

$$\frac{d^2x}{dt^2} = \frac{-k}{m}x$$

- This says that the second derivative of x is proportional to x but has the opposite sign.

SPRING MOTION MODEL

We know two functions with this property, the sine and cosine functions.

- It turns out that all solutions of Equation 3 can be written as combinations of certain sine and cosine functions.

SPRING MOTION MODEL

This is not surprising.

- We expect the spring to oscillate about its equilibrium position.
- So, it is natural to think that trigonometric functions are involved.

GENERAL DIFFERENTIAL EQUATIONS

In general, a differential equation is an equation that contains an unknown function and one or more of its derivatives.

ORDER

The order of a differential equation is the order of the highest derivative that occurs in the equation.

- Equations 1 and 2 are first-order equations.
- Equation 3 is a second-order equation.

INDEPENDENT VARIABLE

In all three equations, the independent variable is called t and represents time.

However, in general, it doesn't have to represent time.

INDEPENDENT VARIABLE**Equation 4**

For example, when we consider the differential equation

$$y' = xy$$

it is understood that y is an unknown function of x .

SOLUTION

A function f is called a solution of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

- Thus, f is a solution of Equation 4 if

$$f'(x) = xf(x)$$

for all values of x in some interval.

SOLVING DIFFERENTIAL EQUATIONS

When we are asked to solve a differential equation, we are expected to find all possible solutions of the equation.

- We have already solved some particularly simple differential equations—namely, those of the form

$$y' = f(x)$$

SOLVING DIFFERENTIAL EQUATIONS

For instance, we know that the general solution of the differential equation $y' = x^3$ is given by

$$y = \frac{x^4}{4} + C$$

where C is an arbitrary constant.

SOLVING DIFFERENTIAL EQUATIONS

However, in general, solving a differential equation is not an easy matter.

- There is no systematic technique that enables us to solve all differential equations.

SOLVING DIFFERENTIAL EQUATIONS

In Section 9.2, though, we will see how to draw rough graphs of solutions even when we have no explicit formula.

We will also learn how to find numerical approximations to solutions.

SOLVING DIFFERENTIAL EQNS. Example 1

Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$

SOLVING DIFFERENTIAL EQNS. Example 1

We use the Quotient Rule to differentiate the expression for y :

$$\begin{aligned} y' &= \frac{(1 - ce^t)(ce^t) - (1 + ce^t)(-ce^t)}{(1 - ce^t)^2} \\ &= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2} \end{aligned}$$

SOLVING DIFFERENTIAL EQNS. Example 1

The right side of the differential equation becomes:

$$\begin{aligned} \frac{1}{2}(y^2 - 1) &= \frac{1}{2} \left[\left(\frac{1 + ce^t}{1 - ce^t} \right)^2 - 1 \right] \\ &= \frac{1}{2} \left[\frac{(1 + ce^t)^2 - (1 - ce^t)^2}{(1 - ce^t)^2} \right] \\ &= \frac{1}{2} \frac{4ce^t}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2} \end{aligned}$$

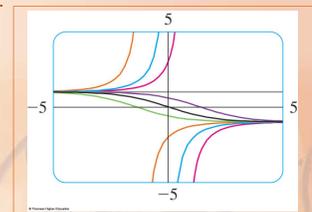
SOLVING DIFFERENTIAL EQNS. Example 1

Therefore, for every value of c , the given function is a solution of the differential equation.

SOLVING DIFFERENTIAL EQNS.

The figure shows graphs of seven members of the family in Example 1.

- The differential equation shows that, if $y \approx \pm 1$, then $y' \approx 0$.
- This is borne out by the flatness of the graphs near $y = 1$ and $y = -1$.



SOLVING DIFFERENTIAL EQNS.

When applying differential equations, we are usually not as interested in finding a family of solutions (the general solution) as we are in finding a solution that satisfies some additional requirement.

- In many physical problems, we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$

INITIAL CONDITION & INITIAL-VALUE PROBLEM

This is called an initial condition.

The problem of finding a solution of the differential equation that satisfies the initial condition is called an initial-value problem.

INITIAL CONDITION

Geometrically, when we impose an initial condition, we look at the family of solution curves and pick the one that passes through the point (t_0, y_0) .

INITIAL CONDITION

Physically, this corresponds to measuring the state of a system at time t_0 and using the solution of the initial-value problem to predict the future behavior of the system.

INITIAL CONDITION**Example 2**

Find a solution of the differential equation

$$y' = \frac{1}{2}(y^2 - 1)$$

that satisfies the initial condition $y(0) = 2$.

INITIAL CONDITION**Example 2**

Substituting the values $t = 0$ and $y = 2$ into the formula from Example 1,

$$y = \frac{1 + ce^t}{1 - ce^t}$$

we get:
$$2 = \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c}$$

INITIAL CONDITION

Example 2

Solving this equation for c ,
we get:

$$2 - 2c = 1 + c$$

This gives $c = \frac{1}{3}$.

INITIAL CONDITION

Example 2

So, the solution of the initial-value
problem is:

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}$$