4.2 The Mean Value Theorem

Preliminary Result:

Rolle's Theorem

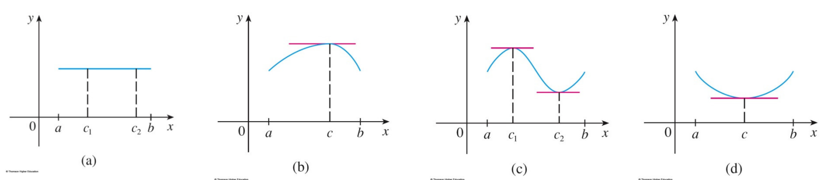
Let $f(x)$ be a function that satisfies the following three hypotheses:

1) f is continuous on the closed interval $[a,b]$.

2) f is differentiable on the open interval (a,b).

3) $f(a)=f(b)$.

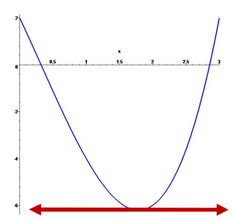
Then there is a number c in (a,b) such that $f'(c)=0$.



Problem:
it doesn't tell us how to find the value of c

Math 103 – Rimmer
4.2 The Mean Value Theorem

Let $f(x)=x^3-x^2-6x+2$. Verify that the function satisfies the three hypotheses of Rolle's Theorem on $[0,3]$. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.



1) f is continuous on the closed interval $[0,3]$ since polynomials are continuous for all values of x.

2) f is differentiable on the open interval $(0,3)$ since polynomials are differentiable for all values of x.

3) $f(0)=2$ and $f(3)=27-9-18+2=2$

So by Rolle's Theorem, there is a number c in $(0,3)$ such that $f'(c)=0$.

$f'(x)=3x^2-2x-6$

$f'(c)=3c^2-2c-6\overset{set}{=}0$

Solve for c.

$$c=\frac{2\pm\sqrt{4-4(3)(-6)}}{2(3)}=\frac{2\pm\sqrt{76}}{6}=\frac{2\pm\sqrt{4\cdot 19}}{6}=\frac{2\pm\sqrt{4}\cdot\sqrt{19}}{6}$$

$$=\frac{2\pm 2\sqrt{19}}{6}=\frac{2}{6}\pm\frac{2\sqrt{19}}{6} \qquad c=\frac{1}{3}\pm\frac{\sqrt{19}}{3}$$

$$c=\frac{1}{3}-\frac{\sqrt{19}}{3}<0 \Rightarrow \text{not in } (0,3) \qquad c=\frac{1}{3}+\frac{\sqrt{19}}{3}\approx 1.79$$

The Mean Value Theorem

Let $f(x)$ be a function that satisfies the following hypotheses:

1) f is continuous on the closed interval $[a,b]$.

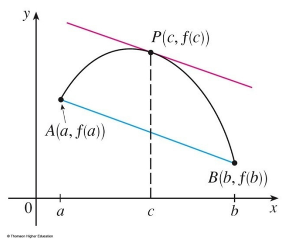
2) f is differentiable on the open interval (a,b).

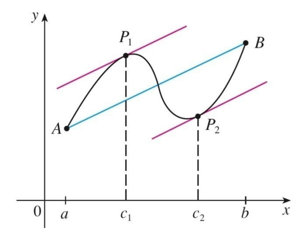
Then there is a number c in (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

or, equivalently, $f'(c)(b-a) = f(b) - f(a)$.

The Mean Value Theorem is a generalization of Rolle's Theorem.

It has the extra condition $f(a) = f(b)$ and looking at the formula above this would make $f'(c) = 0$.





The Mean Value Theorem basically says that there will be at least one place c in (a,b) where the slope of the secant line connecting the endpoints $(a, f(a))$ and $(b, f(b))$ is the same as the slope of the tangent line.

or said another way, there will be a place where the instantaneous rate of change is equal to the average rate of change over an interval

The Mean Value Theorem

Proof:

The slope of the secant line is $\dfrac{f(b)-f(a)}{b-a}$

and a point on the line is $(a, f(a))$.

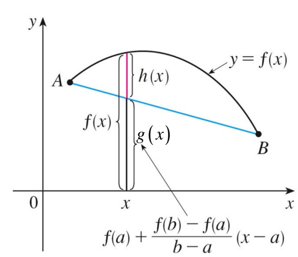
The equation of the secant line is $y - f(a) = \dfrac{f(b)-f(a)}{b-a}(x-a)$.

Equivalently, $y = \dfrac{f(b)-f(a)}{b-a}(x-a) + f(a)$.

We call this function $g(x) = \dfrac{f(b)-f(a)}{b-a}(x-a) + f(a)$

We create a new function $h(x) = f(x) - g(x)$.

$$h(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]$$



We now use Rolle's Theorem on $h(x)$.

1) h is continuous on the closed interval $[a,b]$, its the difference of continuous functions.

2) f is differentiable on the open interval (a,b), its the difference of differentiable functions.

$$h(x)=f(x)-\left[\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right]$$

$$h(a)=f(a)-\left[\frac{f(b)-f(a)}{b-a}(a-a)+f(a)\right]=0$$

$$h(b)=f(b)-\left[\frac{f(b)-f(a)}{b-a}(b-a)+f(a)\right]=f(b)-\left[f(b)-f(a)+f(a)\right]=0$$

3) $h(a)=h(b)$.

So by Rolle's Theorem, there is a number c in (a,b) such that $h'(c)=0$.

$$h(x)=f(x)-\left[\underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{constant}}(x-a)+\underbrace{f(a)}_{\text{constant}}\right]$$

$$h'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$$

so, $h'(c)=f'(c)-\dfrac{f(b)-f(a)}{b-a}=0 \quad \Rightarrow f'(c)=\dfrac{f(b)-f(a)}{b-a}$ our desired result ■

Fall 2008 Final Exam

3. Find the value of c (if any) that satisfies the conclusion of the Mean Value Theorem for the function $f(x)=\dfrac{1}{1+x}$ on the interval $[0,1]$.

A) $\frac{1}{2}$ B) $\frac{1}{4}$ C) $\frac{\sqrt{2}}{2}$ D) $2-\sqrt{2}$ E) $\sqrt{2}-1$ F) no values

$f(x)=\dfrac{1}{1+x}$ f is continuous on $[0,1]$.

$f(x)=(1+x)^{-1}$ $f'(x)=-1(1+x)^{-2}$ $f'(x)=\dfrac{-1}{(1+x)^2}$

f' is defined on $(0,1)$, so f is differentiable on the open interval $(0,1)$.

So by the Mean Value Theorem, there is a number c in $(0,1)$ such that $f'(c)=\dfrac{f(1)-f(0)}{1-0}$.

$f(1)=\frac{1}{2}$ $f(0)=1$ $\dfrac{f(1)-f(0)}{1-0}=\dfrac{\frac{1}{2}-1}{1}=-\dfrac{1}{2}$

$f'(c)=\dfrac{-1}{(1+c)^2}$ So we set $f'(c)=\dfrac{-1}{2}$ and solve for c.

$$\frac{-1}{(1+c)^2}=\frac{-1}{2} \Rightarrow (1+c)^2=2$$
$$\Rightarrow 1+c=\pm\sqrt{2}$$
$c=-1\pm\sqrt{2}$ $c=-1-\sqrt{2}<0 \Rightarrow$ not in $(0,1)$ $c=-1+\sqrt{2}$

The Mean Value Theorem enables us to obtain information about a function from information about its derivative.

<u>4.2 *exercise* #23</u>

If $f(1)=10$ and $f'(x)\geq 2$ for $1\leq x\leq 4$, how small can $f(4)$ possibly be?

1) f is continuous on the closed interval $[1,4]$.

2) f is differentiable on the open interval $(1,4)$.

So by the Mean Value Theorem, there is a number c in $(1,4)$ such that $f'(c)=\dfrac{f(4)-f(1)}{4-1}$.

or, equivalently, $f'(c)(4-1)=f(4)-f(1)$.

$$f(4)=f'(c)(4-1)+f(1)$$

$$f(4)=3f'(c)+10$$

But $f'(c)\geq 2$ for every c in $(1,4)$

$$f(4)\geq 3(2)+10$$

$$f(4)\geq 16$$

$\Rightarrow$ The smallest value for $f(4)=16$.

The Mean Value Theorem is used to prove some basic facts in differential calculus. Here are two such facts:

If $f'(x)=0$ for all x in an interval (a,b), then f is constant on (a,b).

If $f'(x)=g'(x)$ for all x in an interval (a,b), then $f-g$ is constant on (a,b), that is $f(x)=g(x)+c$ where c is a constant.