MATH 104

Solutions to the Practice Problems for Midterm 2

10/26/2017

#1 a) $1 \le x \le 5$, 4 subintervals $\Longrightarrow$ $\Delta x = 1$

$$\int_1^5 \frac{1}{x^2}\,dx \overset{\text{Trapezoid}}{\approx} \frac{1}{2}\left(1 + 2\cdot\frac{1}{4} + 2\cdot\frac{1}{9} + 2\cdot\frac{1}{16} + \frac{1}{25}\right) = \boxed{\frac{3397}{3600}} \ (\cong 0.94)$$

$$\int_1^5 \frac{1}{x^2}\,dx \overset{\text{Simpson}}{\approx} \frac{1}{3}\left(1 + 4\cdot\frac{1}{4} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{16} + 1\cdot\frac{1}{25}\right) = \boxed{\frac{2261}{2700}} \ (\cong 0.83)$$

$\left[\text{cf. } \int_1^5 \frac{1}{x^2}\,dx = \frac{4}{5}\right]$

b) $0 \le x \le 4$ 4 subintervals $\Longrightarrow$ $\Delta x = 1$

$$\int_0^4 x^3\,dx \approx \frac{1}{2}\left(1\cdot 0 + 2\cdot 1 + 2\cdot 8 + 2\cdot 27 + 1\cdot 4^3\right) = \boxed{68}$$

$$\int_0^4 x^3\,dx \approx \frac{1}{3}\left(1\cdot 0 + 4\cdot 1 + 2\cdot 8 + 4\cdot 27 + 1\cdot 4^3\right) = \boxed{64}$$

$\left[\text{cf. } \int_0^4 x^3\,dx = 64\right]$

#2 a) $$\int_0^1 \frac{dx}{x^2} = \lim_{a\to 0_+} \int_a^1 \frac{dx}{x^2} = \lim_{a\to 0_+} -\frac{1}{x}\Big|_a^1 =$$

$$= \lim_{a\to 0_+} -\frac{1}{1} + \frac{1}{a} = \boxed{+\infty} \quad \text{(diverges)}$$

b) $$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{b} + 1 = \boxed{1}$$

c) $$\int_0^\infty \frac{dx}{x^2} = \underbrace{\int_0^1 \frac{dx}{x^2}}_{\text{diverges}} + \underbrace{\int_1^\infty \frac{dx}{x^2}}_{1} = \boxed{+\infty} \quad \text{(diverges)}$$

d) $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} 2\sqrt{x}\Big|_1^b = \lim_{b\to\infty} 2\sqrt{b} - 2 = \infty$

diverges

e) $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0} 2\sqrt{x}\Big|_a^1 = \lim_{a\to 0} 2 - 2\sqrt{a} = \boxed{2}$

#3

a) $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx \le \int_1^\infty \frac{dx}{x^2} = 1$ (by #2 (b))

Hence $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx < \infty$ converges

b) $\int_2^\infty \frac{dx}{\sqrt{x^2-1}} > \int_2^\infty \frac{dx}{\sqrt{x^2}} = \int_2^\infty \frac{dx}{x} = +\infty$ diverges

c) $\int_2^\infty \frac{dx}{\sqrt{x^4-1}}$ $\quad f(x) = \frac{1}{\sqrt{x^4-1}}$ and $g(x) = \frac{1}{x^2}$ are both positive and continuous on $[2,\infty)$ and satisfy $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Since $\int_2^\infty \frac{1}{x^2}\,dx < \infty$ converges, so does $\int_2^\infty \frac{dx}{\sqrt{x^4-1}}$ converge.

(by the Limit Comp. Test)

d) $\int_1^\infty \frac{e^x}{x}\,dx$ diverges because $\lim_{x\to\infty} \frac{e^x}{x} = +\infty$.

e) $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ $\quad f(x) = \frac{1-e^{-x}}{x}$ and $g(x) = \frac{1}{x}$ are both positive and continuous on $[1,\infty)$ and satisfy $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

since $\int_1^\infty \frac{dx}{x}$ diverges, so does $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverge.

(by the Limit Comp Test).

#5 a) $\lim_{n\to\infty} \frac{n+1}{n^3} = \lim_{n\to\infty} \frac{\frac{1}{n^2}^{\to 0} + \left(\frac{1}{n^3}\right)^{\to 0}}{1} = \boxed{0.}$

b) $\lim_{n\to\infty} 3^n = \boxed{\infty}$ (diverges)

c) $\lim_{n\to\infty} \frac{n^3}{3^n} = \boxed{0}$ (L'Hospital 3x)

d) $\lim_{n\to\infty} 3^{1/n} = 3^{\lim_{n\to\infty} 1/n} = 3^0 = \boxed{1.}$

e) $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = \lim_{n\to\infty} \frac{1}{3^n} = \boxed{0.}$

f) $\lim_{n\to\infty} \left(\frac{1}{3}\right)^{1/n} = \left(\frac{1}{3}\right)^{\lim_{n\to\infty} 1/n} = \left(\frac{1}{3}\right)^0 = \boxed{1.}$

g) $\lim_{n\to\infty} \sqrt[n]{3n} = \lim_{n\to\infty} (3n)^{\frac{1}{n}} = \lim_{n\to\infty} \underset{\downarrow\, 1}{3^{\frac{1}{n}}} \cdot \underset{\downarrow\, 1}{n^{\frac{1}{n}}} = \boxed{1.}$

h) $\lim_{n\to\infty} \sqrt[3n]{n} = \lim_{n\to\infty} n^{\frac{1}{3n}} = \lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)^{\frac{1}{3}} = 1^{\frac{1}{3}} = \boxed{1.}$

#6 $\ln\left[\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^{bn}\right] = \lim_{n\to\infty} \ln\left[\left(1+\frac{a}{n}\right)^{bn}\right] = \lim_{n\to\infty} bn \ln\left(1+\frac{a}{n}\right)$

$= \lim_{n\to\infty} b \frac{\ln\left(1+\frac{a}{n}\right)}{\frac{1}{n}} \underset{\text{L'Hospital}}{\overset{}{=}} \lim_{n\to\infty} b. \frac{\frac{1}{1+\frac{a}{n}^{\to 0}}\left(-\frac{a}{n^2}\right)}{-\frac{1}{n^2}} = ab.$

2

$$\Rightarrow \boxed{\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{bn} = e^{ab}}.$$

#4 a) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{+\infty} C x^2 e^{-4x}\,dx = C\cdot\lim_{b\to\infty}\int_0^b x^2 e^{-4x}\,dx$

Parts $\displaystyle \overset{\curvearrowright}{=} C\cdot\lim_{b\to\infty} -\frac{e^{-4x}}{4}x^2 - \frac{e^{-4x}}{8}x - \frac{e^{-4x}}{32}\Bigg|_0^b$

$$= C\cdot\lim_{b\to\infty} \underbrace{-\frac{e^{4b}}{4}b^2 - \frac{e^{-4b}}{8}b - \frac{e^{-4b}}{32}}_{\to 0} + \frac{1}{32} = \frac{C}{32}$$

Since need $\int_{-\infty}^{\infty} f(x)\,dx = 1$, it follows that $\boxed{C = 32}$

b) $\displaystyle \mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{+\infty} 32x^3 e^{-4x}\,dx = \lim_{b\to\infty}\int_0^b 32x^3e^{-4x}\,dx$

Parts $\displaystyle \overset{\curvearrowright}{=} \lim_{b\to\infty} \frac{3}{4} \underbrace{- \frac{3}{4}e^{-4b} - 3be^{-4b} - 6b^2e^{-4b} - 8b^3e^{-4b}}_{\to 0} = \boxed{\frac{3}{4}}$

#17 Note: $(n!)^2 = n^2(n-1)^2(n-2)^2(n-3)^2\cdots 3^2\cdot 2^2\cdot 1^2$

$$= \underbrace{n\cdot 1}_{\geq n}\cdot\underbrace{(n-1)\cdot 2}_{\geq n}\cdot\underbrace{(n-2)\cdot 3}_{\geq n}\cdots\underbrace{3\cdot(n-2)}_{\geq n}\cdot\underbrace{2(n-1)}_{\geq n}\cdot\underbrace{1\cdot n}_{\geq n}$$

$$\geq n^n$$

because $(n-k)\cdot(k+1) = nk + n - k^2 - k = n + k(n-k-1) \geq n$ for all $0 \leq k < n$.

Thus $\boxed{n! \geq n^{n/2}}$ for all $n \geq 1$.

Therefore

$$\lim_{n\to\infty} \sqrt[n]{n!} \geq \lim_{n\to\infty} \sqrt[n]{n^{n/2}} = \lim_{n\to\infty} \sqrt{n} = +\infty \quad \underline{\text{diverges.}}$$

#8 $$\frac{n!}{2^n} = \frac{n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1}{2 \quad 2 \quad 2 \ldots 2 \cdot 2 \cdot 2}$$

(if $n>2$) $$> \frac{3 \cdot 3 \cdot 3 \ldots 3 \; 2 \; 1}{2 \quad 2 \quad 2 \ldots 2 \; 2 \; 2} = \frac{1}{2}\left(\frac{3}{2}\right)^{n-2}$$

Thus $$\lim_{n\to\infty} \frac{n!}{2^n} > \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^{n-2} = +\infty \quad \left(\tfrac{3}{2}>1\right) \quad \underline{\text{diverges.}}$$

#9 Up to replacing $a>0$ by the smallest integer larger than a, <u>suppose a is an integer</u>. Just like in #8, provided $n>a$, then

$$\frac{n!}{a^n} = \frac{n \cdot (n-1)(n-2) \ldots (a+1) \cdot a \ldots 3 \cdot 2 \cdot 1}{a \quad a \quad a \ldots a \cdot a \ldots a \cdot a \cdot a}$$

($n>a$) $$> \frac{(a+1)(a+1)(a+1) \ldots (a+1)\, a \ldots 3 \cdot 2 \cdot 1}{a \cdot a \cdot a \ldots a \quad a \ldots a \; a \; a}$$

$$= \left(\frac{a+1}{a}\right)^{n-a} \cdot \frac{a!}{a^a}$$

Thus $$\lim_{n\to\infty} \frac{n!}{a^n} > \lim_{n\to\infty} \left(\frac{a+1}{a}\right)^{n-a} \cdot \frac{a!}{a^a} = +\infty \quad \left(\tfrac{a+1}{a}>1\right) \quad \underline{\text{diverges.}}$$

#10

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \boxed{\frac{1}{3}}$

Geometric

$a = \frac{1}{4}$, $r = \frac{1}{4}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{7}{2^n} = \frac{7/2}{1-1/2} = \frac{7/2}{1/2} = \boxed{7}$

Geometric

$a = \frac{7}{2}$, $r = \frac{1}{2}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{n^2+4} = +\infty$ <u>diverges</u> because $\displaystyle\lim_{n\to\infty} \frac{6^n}{n^2+4} = +\infty \ (\neq 0)$

("n^{th} term test")

d) Telescoping: $\displaystyle S_n = \sum_{k=1}^{n} \tan k - \tan(k+1)$

$= (\tan 1 - \cancel{\tan 2}) + (\cancel{\tan 2} - \cancel{\tan 3}) + \cancel{\cdots} + \cancel{\tan n} - \tan(n+1)$

$= \tan 1 - \tan(n+1) \leftarrow$ (this sequence does not converge as $n \to \infty$)

$\displaystyle\sum_{n=1}^{\infty} \tan n - \tan(n+1) = \lim_{n\to\infty} S_n$ does not exist; <u>diverges</u>.

e) Telescoping: $\displaystyle S_n = \sum_{k=1}^{n} \arccos\left(\frac{1}{k+1}\right) - \arccos\left(\frac{1}{k+2}\right)$

$= \left(\arccos\frac{1}{2} - \cancel{\arccos\frac{1}{3}}\right) + \left(\cancel{\arccos\frac{1}{3}} - \arccos\frac{1}{4}\right) + \cdots + \left(\cancel{\arccos\frac{1}{n+1}} - \arccos\frac{1}{n+2}\right)$

$= \arccos\frac{1}{2} - \arccos\frac{1}{n+2}$

$= \frac{\pi}{3} - \arccos\frac{1}{n+2}$.

$$\sum_{n=1}^{\infty} \arccos\frac{1}{n+1} - \arccos\frac{1}{n+2} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{\pi}{3} - \arccos\frac{1}{n+2} = \frac{\pi}{3} - \frac{\pi}{2} = \boxed{-\frac{\pi}{6}}$$

#11

a) $$\sum_{n=1}^{\infty} \frac{n^2}{n^4+1} < \sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \quad \left(\begin{matrix}p\text{-series}\\ p=2\end{matrix}\right)$$

Direct Comp. Test

Converges.

b) $a_n = \dfrac{n^5}{n^6+2n^3+1}$ Let $b_n = \dfrac{1}{n}$. Then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 1$,

and since $\sum_{n=1}^{\infty} b_n = +\infty$ (Harmonic Series), also $\sum_{n=1}^{\infty} a_n = +\infty$ diverges by the Limit Comp. Test.

c) $\sum_{n=2}^{\infty} \dfrac{3^{n+2}}{\ln n} = +\infty$ diverges because $\lim_{n\to\infty} \dfrac{3^{n+2}}{\ln n} = +\infty (\neq 0)$

("nth term Comp. Test")

d) $a_n = \left(\dfrac{4n+1}{2n-5}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \dfrac{4n+1}{2n-5} = 2 > 1$: $\sum_{n=1}^{\infty} a_n$ diverges by Root Test.

e) $a_n = \dfrac{4^n}{3^n n^n}$ $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{4^{n+1}}{3^{n+1}(n+1)^{n+1}} \cdot \dfrac{3^n \cdot n^n}{4^n}$

$\sum_{n=1}^{\infty} a_n$ Converges by the Ratio Test.

$$= \lim_{n\to\infty} \frac{4}{3} \underbrace{\left(\frac{n}{n+1}\right)^n}_{\downarrow\ \frac{1}{e}} \cdot \underbrace{\frac{1}{n+1}}_{\downarrow\ 0} = 0 < 1$$

f) $a_n = \dfrac{n+2}{4^n}$ $\quad \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{n+3}{4^{n+1}} \cdot \dfrac{4^n}{n+2} = \dfrac{1}{4} < 1$

So $\sum_{n=1}^{\infty} a_n$ converges by the Ratio Test.

#12 a) Amount of drug in the body from previous day is 10% of the dosage of 10 mg. The dosage from 2 days prior has decayed by 10% twice, so there is $100 \cdot (0.1)^2 = 1$ mg left from that dose. Similarly, amount left from the first dose 3 days ago is $100 \cdot (0.1)^3 = 0.1$ mg. Thus, the total amount is $10 + 1 + 0.1 = \underline{11.1 \text{ mg}}$.

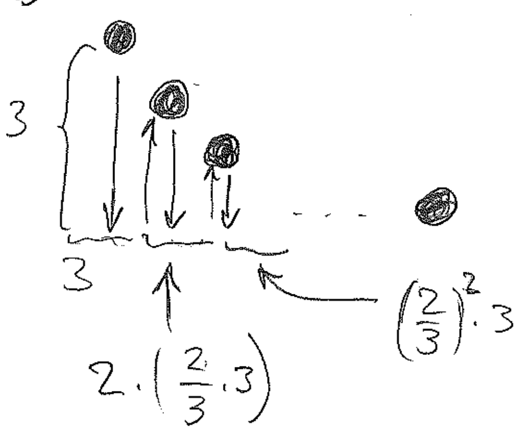
b) Following above pattern, amount in is:

$$M = \underbrace{100(0.1)}_{\text{yesterday}} + \underbrace{100 \cdot (0.1)^2}_{\text{2 days ago}} + \dots + \underbrace{100 \cdot (0.1)^n}_{\text{n days ago}} + \dots$$

$$= \sum_{n=1}^{\infty} 100 \cdot (0.1)^n \overset{\text{Geometric}}{=} \frac{10}{1 - 1/10} = \frac{10}{9/10} = \boxed{\frac{100}{9} \text{ mg}}$$

Geometric $a = 10,\ r = \frac{1}{10}$

#13



(up & down) (2/3 of 3m) (up & down) ($\frac{2}{3}$ of $\frac{2}{3}$ of 3)

$$\text{Total dist.} = \underbrace{3}_{\substack{\text{first drop}\\\text{(only down)}}} + \underbrace{2 \cdot \frac{2}{3} \cdot 3}_{\text{second drop}} + \underbrace{2\left(\frac{2}{3}\right)^2 \cdot 3}_{\text{third drop}} + \dots$$

$$= 3 + 2 \cdot \sum_{n=1}^{+\infty} \left(\frac{2}{3}\right)^n \cdot 3$$

Geometric $a = 2,\ r = 2/3$

$$= 3 + 2 \cdot \frac{2}{1 - 2/3} = 3 + \frac{4}{1/3} = \boxed{15 \text{ m}}$$