Introduction to Elliptic PDEs

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1 Introduction

For the next several weeks we will be looking at elliptic equations of the form

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u + c(x) u = f(x) \text{ in } \Omega. \tag{1}$$

Now let's discuss what the parts of this equation mean. In the process we will discuss some of the notation for this course. Here:

- Let U be an open set in $\mathbb{R}^n$. We let $C^0(U)$ denote the space of all continuous real-valued functions on U. For $k = 1, 2, 3, \ldots$, we let $C^k(U)$ denote the space of all real-valued functions on U for which all derivatives up to order k exist and are continuous on U. We let C^∞ denote the space of all real-valued functions on U for which all derivatives exist up to all orders.

- $D_i = \partial/\partial x_i$ is the standard partial derivative in the x_i-direction, $D_{ij} = \partial^2/\partial x_i \partial x_j$ is the second order mixed partial derivative in the x_i and x_j directions. In particular $D_i u$ and $D_{ij} u$ are derivatives of u. Note that more generally we will use multi-index notation where

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

  for integers $\alpha_i \geq 0$ and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

- Ω is a *domain*, i.e. a connected open set, in $\mathbb{R}^n$.

- We say Ω is a C^k domain for $k \geq 1$ if for every point y in $\partial\Omega = \overline{\Omega} \setminus \Omega$, there exists a $\delta > 0$ and C^k diffeomorphism $\Psi : B_\delta(y) \to \mathbb{R}^n$ such that

$$\Psi(\Omega \cap B_\rho(y)) = \{x = (x_1, x_2, \ldots, x_n) \in B_1(0) : x_n > 0\},$$
$$\Psi(\partial\Omega \cap B_\rho(y)) = \{x = (x_1, x_2, \ldots, x_n) \in B_1(0) : x_n = 0\}.$$

- $u \in C^2(\Omega)$.

- $a^{ij}, b^i, c : \Omega \to \mathbb{R}$ are functions on Ω, called the *coefficients* of L. We shall assume WLOG that $a^{ij} = a^{ji}$.

- $f : \Omega \to \mathbb{R}$ is also a function on Ω.

- The L is the linear map, called an *operator*, from $C^2(\Omega)$ to real-valued functions on Ω given by

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u + c(x) u.$$

- Ellipticity condition: Recall that

$$\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2 \text{ for every } x \in \Omega,\ \xi \in \mathbb{R}^n$$

  where $\lambda(x)$ and $\Lambda(x)$ are the minimum and maximum eigenvalues respectively of the $n \times n$ symmetric matrix $(a^{ij})_{i,j=1,\ldots,n}$. We say that the equation (1) or the operator L is

  (i) *elliptic* if $\lambda(x) > 0$ for all $x \in \Omega$,

  (ii) *strictly elliptic* if $\lambda(x) \geq \lambda_0 > 0$ for all $x \in \Omega$ and some constant $\lambda_0 > 0$, or

  (iii) *uniformly elliptic* if $\lambda(x) > 0$ for all $x \in \Omega$ and $\sup_{x \in \Omega} \Lambda(x)/\lambda(x) < \infty$.

  Note that we can always assume L is strictly elliptic since if L is elliptic then $\frac{1}{\lambda} L$ is strictly elliptic.

We will often write (1) in Einstein notation,

$$Lu = a^{ij}(x) D_{ij} u + b^i(x) D_i u + c(x) u = f(x) \text{ in } \Omega,$$

where the Σ denoting sums are omitted and repeated indices denote sums.

Related to equations of this form is the *Dirichlet problem*

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned} \tag{2}$$

where L is the elliptic operator from above, $\partial\Omega = \overline{\Omega} \setminus \Omega$ is the frontier or *boundary* of Ω, $f : \Omega \to \mathbb{R}$ is a function, and $\varphi : \partial\Omega \to \mathbb{R}$ is a function called the *boundary data*.

When reading theorems in this class, it will be important to know:

- The topological properties of Ω: Is Ω an open set or a domain? Is Ω bounded or unbounded?

- The regularity of Ω, i.e. is Ω a C^k domain?

- The starting regularity of u both in Ω and up to the boundary of Ω. For example, we might assume $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$.

- Ellipticity condition: Is L elliptic, strictly elliptic, or uniformly elliptic?

- Regularity of the coefficients a^{ij}, b^i, and c, i.e. whether the coefficients are bounded, continuous, in C^k, etc, and any bounds on the coefficients.

- Sign of c.

- Regularity assumptions on f and φ, i.e. whether they are bounded, continuous, in C^k, etc.

2 Example: Poisson equation

The *Poisson equation* is an elliptic equation of the form

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = f \text{ in } \Omega.$$

It is obvious that Δ is a uniformly elliptic operator as $\lambda = \Lambda = 1$ on Ω. When $f = 0$ on Ω, we obtain the *Laplace equation*

$$\Delta u = 0 \text{ in } \Omega.$$

Solutions to the Laplace equation are called *harmonic functions*. A function u is harmonic if and only if u minimizes the energy functional

$$E(u) = \frac{1}{2} \int_{\Omega} |Du|^2$$

with the constraint that $u = \varphi$ on $\partial\Omega$ for some continuous real-valued function φ on $\partial\Omega$. (See Example Sheet 1)

3 Three main questions

In this course, there are three main questions we want to ask about our elliptic operator L and the Dirichlet problem for L:

1. Uniqueness: Is there at most one solution to the Dirichlet problem (2)?
2. Existence: Is there at least one solution to the Dirichlet problem (2)?
3. Regularity: Suppose u is a solution to either (1) or (2). Then what can we say about the regularity of u, i.e. the extent to which we can take derivatives of u and still obtain a continuous (or L^2) function?

An answer to these questions, we need to build up the following tools:

- Maximum principle: A solution to $Lu = 0$ in Ω obtains its maximum value on the boundary of Ω.
- A priori estimates for the Dirichlet problem: $\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |\varphi| + C \sup_{\Omega} |f|$.
- Schauder estimates for the Dirichlet problem: $\|u\|_{C^{2,\mu}(\Omega)} \leq C(\sup_{\Omega} |u| + \|f\|_{C^{0,\mu}(\Omega)} + \|\varphi\|_{C^{2,\mu}(\partial\Omega)})$.

Note that these are rough statements of the tools and that there are some important hypotheses that I am not mentioning at the moment. We will spend a number of lectures building up the tools and a number of more lectures addressing the answers to the three main questions above from there.

4 First theorem: Weak Maximum Principle

Theorem 1. *(Weak Maximum Principle) Let Ω be a bounded open set in $\mathbb{R}^n$. Suppose $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu \geq 0 \text{ in } \Omega$$

for some functions a^{ij}, b^i, and c on Ω. Suppose L is an elliptic operator and

$$\sup_\Omega \frac{|b^i|}{\lambda} + \sup_\Omega \frac{|c|}{\lambda} < \infty.$$

Suppose $c \leq 0$ on Ω. Then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+,$$

where $u^+(x) = \max\{u(x), 0\}$ at each $x \in \Omega$. Moreover, if $Lu = 0$ in Ω, then

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u|. \tag{3}$$

To see how the $Lu = 0$ in Ω case follows from the more general case of $Lu \geq 0$ in Ω, observe that since $Lu = 0$ in Ω, u^+ attains its maximum value on the boundary of Ω. Since $L(-u) = 0$ in Ω, we also get a minimum principle that $u^-(x) = \min\{u(x), 0\}$ attains its minimum value on the boundary of Ω. Thus in effect u attains its maximum and minimum values on the boundary of Ω and (3) follows.

Corollary 1. *(Uniqueness of Solutions to the Dirichlet Problem) Let Ω be a bounded open set in $\mathbb{R}^n$. Consider the Dirichlet problem*

$$\begin{aligned} Lu = a^{ij} D_{ij} u + b^i D_i u + cu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

for some functions a^{ij}, b^i, c, and f on Ω and $\varphi \in C^0(\partial\Omega)$ such that L is an elliptic operator,

$$\sup_\Omega \frac{|b^i|}{\lambda} + \sup_\Omega \frac{|c|}{\lambda} < \infty,$$

and $c \leq 0$ in Ω. Then there is at most one solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ to the Dirichlet problem (i.e. there may be no solution or a unique solution but there cannot be two or more solutions).

Proof. Suppose u_1 and u_2 are two solutions to the Dirichlet problem. Then $w = u_1 - u_2$ satisfies

$$\begin{aligned} Lw &= 0 \text{ in } \Omega, \\ w &= 0 \text{ on } \partial\Omega. \end{aligned}$$

By the Weak Maximum Principle,

$$\sup_\Omega |w| \leq \sup_{\partial\Omega} |w| = 0.$$

Therefore $w = u_1 - u_2 = 0$ on $\overline{\Omega}$, i.e. $u_1 = u_2$ on $\overline{\Omega}$. (Note how uniqueness of solutions to the Dirichlet problem correspond to there being a solution to $Lw = 0$ in Ω and $w = 0$ on $\partial\Omega$ other than the trivial solution $w \equiv 0$.) □