GROUP THEORY PRACTICE PROBLEMS 1

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Contents

1. Basic definition

Problem 1.1. *Prove that if G is an abelian group, then for all $a, b \in G$ and all integers n, $(a \cdot b)^n = a^n \cdot b^n$.*

Problem 1.2. *If G is a group such that $(a \cdot b)^2 = a^2 \cdot b^2$ for all $a, b \in G$, show that G must be abelian.*

Problem 1.3. *If G is a finite group, show that there exists a positive integer N such that $a^N = e$ for all $a \in G$.*

Problem 1.4.
(1) *If the group G has three elements, show it must be abelian.*
(2) *Do part (1) if G has four elements.*
(3) *Do part (2) if G has four elements*

Problem 1.5. *Show that if every element of the group G is its own inverse, then G is abelian.*

Problem 1.6. *If G is a group of even order, prove it has an element $a \neq e$ satisfying $a^2 = e$.*

Problem 1.7. *For any $n > 2$ construct a non-abelian group of order $2n$. (Hint: imitate the relations in S_3.)*

2. Subgroups

Problem 2.1. *If G has no nontrivial subgroups, show that G must be finite of prime order.*

Problem 2.2.
(1) *If H is a subgroup of G, and $a \in G$. Let $aHa^{-1} = \{aha^{-1} | h \in H\}$. Show that aHa^{-1} is a subgroup of G.*
(2) *If H is finite, what is the order of aHa^{-1}?*

Problem 2.3. *Write out all the right cosets of H in G where*
(1) *$G = (a)$ is a cyclic group of order 10 and $H = (a^2)$ is the subgroup of G generated by a^2.*

(2) *G as in part (1), $H = (a^5)$ is the subgroup of G generated by a^5.*

Problem 2.4. *If $a \in G$, define $N(a) = \{x \in G | xa = ax\}$. Show that $N(a)$ is a subgroup of G. $N(a)$ is usually called the* **normalizer** *or* **centralizer** *of a in G.*

Problem 2.5. *If H is a subgroup of G, then by the* **centralizer** *$C(H)$ of H we mean the set $\{x \in G | xh = hx \ \forall h \in H\}$. Prove that $C(H)$ is a subgroup of G.*

Problem 2.6. *The* **center** *$Z(G)$ of a group G is defined by $Z(G) = \{z \in G | zx = xz \ \forall x \in G\}$. Prove that $Z(G)$ is a subgroup of G. Can you recognize Z as $C(T)$ for some subgroup T of G?*

Problem 2.7. *If H is a subgroup of G, let $N(H) = \{a \in G | aHa^{-1} = H\}$. Prove that*

(1) *$N(H)$ is a subgroup of G.*
(2) *$H \subset N(H)$.*

We call $N(H)$ the **normalizer** *of H in G.*

Problem 2.8. *If $a \in G$ and $a^m = e_G$, prove that the order of a divides m.*

3. Homomorphisms

Problem 3.1. *Let G be a finite abelian group of order $ord(G)$ and suppose the integer n is relatively prime to $ord(G)$. Prove that every $g \in G$ can be written as $g = x^n$ with $x \in G$. (*Hint*: Consider the mapping $\phi : G \to G$ defined by $\phi(y) = y^n$, and prove this mapping is an isomorphism of G onto G.*

Problem 3.2. *Let G be the dihedral group defined as $\{x, y | x^2 = e, \ y^n = e, \ xy = y^{-1}x\}$. Prove*

(1) *The subgroup $N = \{e, y, y^2, \cdots, y^{n-1}\}$ is normal in G.*
(2) *That $G/N \simeq W$, where $W = \{1, -1\}$ is the group under the multiplication of the real numbers.*

Problem 3.3. *Prove that a group of order 9 is abelian.*

Problem 3.4. *If G is a non-abelian group of order 6, prove that $G \simeq S_3$.*

Problem 3.5. *If G is abelian and if N is any subgroup of G, prove that G/N is abelian*

Problem 3.6. *Let G be the group of all nonzero complex numbers under multiplication and let $\bar{G}$ be the group of all real 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where not both a and b are 0, under matrix multiplication. Show that G and $\bar{G}$ are isomorphic by exhibiting an isomorphism of G onto $\bar{G}$.*

References

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