THE NOVIKOV CONJECTURE FOR MAPPING CLASS GROUPS AS A COROLLARY OF HAMENSTÄDT'S THEOREM

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Let $\mathrm{Mod}(S)$ denote the mapping class group of a finite type surface. This short note combines theorems of Kato and Hamenstädt to show

Corollary 1. *The higher signatures of Mod(S) are oriented homotopy invariants. In other words, the Novikov conjecture is true for Mod(S).*

The corollary virtually follows immediately from the following theorems. The first is due to Kato.

Theorem 2. [4, Thm.0.1] *Let Γ be a torsion free finitely generated group which admits a proper combing of bounded multiplicity. Then the higher signatures of Γ are oriented homotopy invariants.*

We will not define a "proper combing of bounded multiplicity". Instead we simply note that any "quasi-geodesically bicombable group can admit a structure of proper combing [*sic*] of strictly bounded multiplicity [4, Ex.2.1,pg.68]." The second theorem is due to Hamenstädt.

Theorem 3. [1, Sec.6] *The mapping class group Mod(S) admits a quasigeodesic bicombing.*

In Section 6 of [1], Hamenstädt builds a quasigeodesic bicombing of the train track complex (notated $\mathcal{TT}$), which she shows to be quasi-isometric to $\mathrm{Mod}(S)$.

Remark 4. *The proof of Corollary 1 rests entirely on Kato's Thorem 2 and Hamenstädt's Theorem 3. It is hoped that this note will bring further attention to these two important results.*

We now recall the statement of the Novikov conjecture. Let M be a closed oriented smooth n-manifold with fundamental class $[M] \in H_n(M;\mathbb{Q})$. Let $L_M \in H(M;\mathbb{Q})$ denote the Hirzebruch L-class of M, which is defined as a (fixed) power series in the Pontryagin classes of M. Let Γ be a discrete group with an Eilenberg-Maclane space $B\Gamma$. For a $u \in H(B\Gamma;\mathbb{Q})$ and a continuous $f : M \longrightarrow B\Gamma$ define the higher signature

$$\langle L_M \cup f^*u, [M]\rangle \in \mathbb{Q}.$$

The higher signature defined by u and f is an oriented homotopy invariant if: for any closed oriented smooth manifold N with fundamental class $[N]$ and a homotopy equivalence $h : N \longrightarrow M$ taking $[N]$ to $[M]$ we have the equality

$$\langle L_M \cup f^*u, [M]\rangle = \langle L_N \cup (f \circ h)^*u, [N]\rangle. \qquad (\dagger)$$

For more information on the Novikov conjecture see [5].

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A finite presentation for $\mathrm{Mod}(S)$ was given by Hatcher-Thurston [2]. Note also the following well known fact.

Proposition 5. [3, Ch.5.5] *The mapping class group contains a finite index normal torsion free subgroup* Λ.

Combining Theorem 2, Theorem 3, and the above proposition establishes the Novikov conjecture for Λ. It remains only to extend the result to all of $\mathrm{Mod}(S)$.

Define $\Gamma := \mathrm{Mod}(S)$ and let M, N, f, h, and u be as above. We must establish equation (†). Let $\widetilde{M}$, $\widetilde{N}$, $\widetilde{f}$, and $\widetilde{h}$ denote the lifts of M,N, f, and h respectively to the finite covering spaces corresponding to $(f_*)^{-1}(\Lambda) \leq \pi_1(M)$, where $f_* : \pi_1(M) \longrightarrow \Gamma$ is the induced map on π_1. We will use π to denote either of the covering maps $\widetilde{M} \longrightarrow M$, $\widetilde{N} \longrightarrow N$, or $B\Lambda \longrightarrow B\Gamma$. Applying the Novikov conjecture for Λ to these lifts yields the equation

$$\langle L_{\widetilde{M}} \cup \widetilde{f}^*(\pi^* u), [\widetilde{M}] \rangle = \langle L_{\widetilde{N}} \cup (\widetilde{f} \circ \widetilde{h})^*(\pi^* u), [\widetilde{N}] \rangle.$$

The naturality of the pairing implies

$$\begin{aligned} \langle \pi^* L_M \cup \widetilde{f}^*(\pi^* u), [\widetilde{M}] \rangle &= \langle \pi^* L_M \cup \pi^*(f^* u), [\widetilde{M}] \rangle \\ &= \langle L_M \cup f^* u, \pi_*[\widetilde{M}] \rangle = k \cdot \langle L_M \cup f^* u, [M] \rangle, \end{aligned}$$

where k is the index of $(f_*)^{-1}(\Lambda)$ in $\pi_1(M)$. Similarly

$$\begin{aligned} \langle \pi^* L_N \cup (\widetilde{f} \circ \widetilde{h})^*(\pi^* u), [\widetilde{N}] \rangle &= \langle \pi^* L_N \cup \pi^*((f \circ h)^* u), [\widetilde{N}] \rangle \\ &= \langle L_N \cup (f \circ h)^* u, \pi_*[\widetilde{N}] \rangle \\ &= k \cdot \langle L_N \cup (f \circ h)^* u, [N] \rangle. \end{aligned}$$

By naturality $L_{\widetilde{M}} = \pi^* L_M$ and $L_{\widetilde{N}} = \pi^* L_N$. Therefore

$$\begin{aligned} k \cdot \langle L_M \cup f^* u, [M] \rangle &= \langle L_{\widetilde{M}} \cup \widetilde{f}^*(\pi^* u), [\widetilde{M}] \rangle \\ &= \langle L_{\widetilde{N}} \cup (\widetilde{f} \circ \widetilde{h})^*(\pi^* u), [\widetilde{N}] \\ &= k \cdot \langle L_N \cup (f \circ h)^* u, [N] \rangle. \end{aligned}$$

This establishes equation (†) and completes the proof.

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