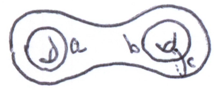
Lecture 6

May 20, 2009

It's worth noting the geometric meaning of small distances in the curve graph:

$d(b,c) = 1 \iff$ b + c are disjoint

$d(b,c) = 2 \iff \exists$ a disjoint from b + c



$d(b,c) > 2 \iff$ $M-(b\cup c)$ is a union of disks and annuli. (If $\partial M = \emptyset$ then $M-(b\cup c)$ is only disks. The annuli always have one ~~comp~~ boundary component lying in ∂M.)

Seeing the difference between distance 7 and 70 is not so easy.

The next goal is to prove that the curve complex has infinite diameter. More precisely, we'll show the curve graph has infinite diameter. The argument here is due to T. Kobayashi "Heights of simple loops and pseudo-Anosov homeomorphisms" (Prop 2.2). The argument requires several preliminary facts from Teichmüller theory. Let me state them first, then ~~prove the~~ give Kobayashi's argument. We'll go back and fill in the facts afterward, if we have time.

FACTS:

1. The space of projective measured foliations $\mathbb{P}\mathcal{MF}$ is compact. (In fact it's a sphere, but we won't need this.)
2. The intersection number $i : \mathcal{S} \times \mathcal{S} \to \{0, 1, 2, \dots\}$ extends to a continuous fct. $i : \mathcal{MF} \times \mathcal{MF} \to [0, \infty)$.
3. Recall the definition of a pseudo-Anosov homeomorphism. We will need the existence of at least one pseudo-Anosov homeo with unstable ~~lamination~~ foliation $\mathcal{F}$. Moreover, we'll need the following fact that we have not seen at all previously:

$$i(\mu, \mathcal{F}) = 0 \Rightarrow \mu = \mathcal{F}.$$

Let $\mathcal{F}_c$ be the measured foliation determined by $c \in \mathcal{S}$. From the def'n of pseudo-Anosov it follows that $\mathcal{F}$ has no closed leaves. In particular, $i(\mathcal{F}_c, \mathcal{F}) > 0$, + $\mathcal{F}_c \neq \mathcal{F}$.

4. Recall there is an embedding $\mathcal{S} \hookrightarrow \mathbb{P}\mathcal{MF}$. We'll use that the image is dense.

Thm: The curve graph of M has infinite diameter.

Pf: Pick $c \in \mathcal{S}$ and let $Z_n \subset \mathbb{P}\mathcal{MF}$ be the (compact) closure of the set $\{b \in \mathcal{S} \mid d(b, c) \leq n\}$.

As above, let $\mathcal{F}$ be the unstable foliation of some ψ-Anosov homeomorphism. By the remarks in Fact 3 above, $Z_0 = \{c\} \neq \{\mathcal{F}\}$. Suppose by induction that $Z_k \cap \{\mathcal{F}\} = \emptyset \;\forall\; k < n$. In search of a contradiction assume $\mathcal{F} \in Z_n$. Then $\exists\; \{b_j\} \subset \mathcal{S}$ s.t. $d(b_j, c) = n$ and $b_j \to \mathcal{F}$ in $\mathbb{P}\mathcal{MF}$. Also $\exists\; \{a_j\} \subset \mathcal{S}$ s.t. $d(a_j, c) = n-1$ + $d(b_j, c_j) = 1$, implying $i(b_j, c_j) = 0$. Up to subsequence $a_j \to \lambda \in Z_{n-1}$. By continuity

$$0 = i(a_j, b_j) \to i(\lambda, \mathcal{F}) \quad \Rightarrow \quad \lambda = \mathcal{F}.$$

Contradiction. $\therefore \;\forall\; n$, $\mathcal{F} \notin Z_n$.

By Fact 4, $\exists$ scc $c_n \in \mathcal{S}$ s.t. $c_n \in \mathbb{P}\mathcal{MF} - Z_n$.

Then $d(c, c_n) > n$. $\boxtimes$

A overview of the general Masur-Minsky machinery

To state the general bounded geodesic image theorem we must define subsurface projections.

Suppose $X \subset M$ satisfies:

- M is a compact oriented connected surface with (possibly empty) boundary. ~~not~~ Assume $M \notin \{$ , , , ,  $\}$.

  (For $M =$  join scc's with intersection number 1.
  For $M =$  join scc's with intersection number 2.)

- X is a connected proper compact subsurface s.t. $X \hookrightarrow M$ induces an injection $\pi_1 X \hookrightarrow \pi_1 M$,

  X is not freely homotopic into ∂M,

  $X \notin \{$ ,  $\}$. (X also is not , )

  For simplicity, let's assume $X \neq$ . (This case is pesky.)

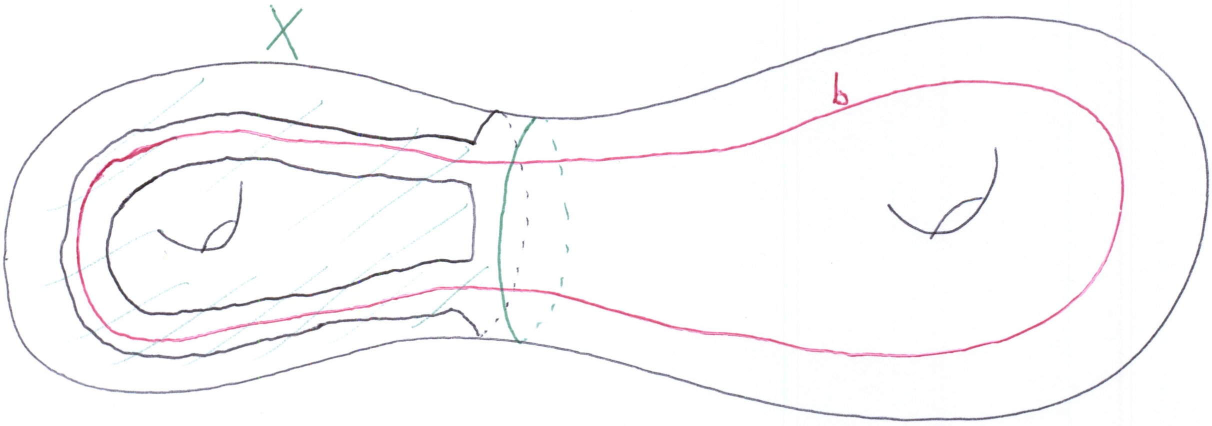
Define projection

$$\pi_X : \mathcal{C}(M) \longrightarrow \left(\begin{matrix}\text{subsets of}\\ \mathcal{C}(X)\end{matrix}\right)$$

($\mathcal{C}$ indicates the curve graph.)

Pick $b \in \mathcal{C}(M)$. Assume we chose b to intersect ∂X minimally in its homotopy class. If $b \cap X = \emptyset$ then $\pi_X(b) = \emptyset$. If $b \subset X$ then $\pi_X(b)$ is simply $\{b\}$, thought of as a scc of X. Otherwise let $\{b_1, b_2, \dots, b_n\}$ be the components of $b \cap X$. For each b_i let $X_i \subset \partial X$ be the components of ∂X intersecting b_i, and consider a small closed neighborhood

K of $b_i \cup X_i$. Let S_i be the components of ∂K that are homotopically nontrivial and not homotopic into ∂X. S_i must be nonempty. Let $\pi_X(b) = \bigcup S_i$. The diameter of $\pi_X(b)$ is ≤ 2.



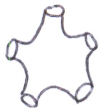
Note that some components of S_i may be homotopic in X, as in the picture's example.

[Bounded geodesic image thm (Masur-Minsky): Let $g \subset \mathcal{C}(M)$ be a (possibly infinite) geodesic such that $\pi_X(v) \neq \emptyset$ for every vertex v of g. Then there is a constant $D(M)$ such that the set $\bigcup_{v \in g} \pi_X(v)$ has diameter at most D.

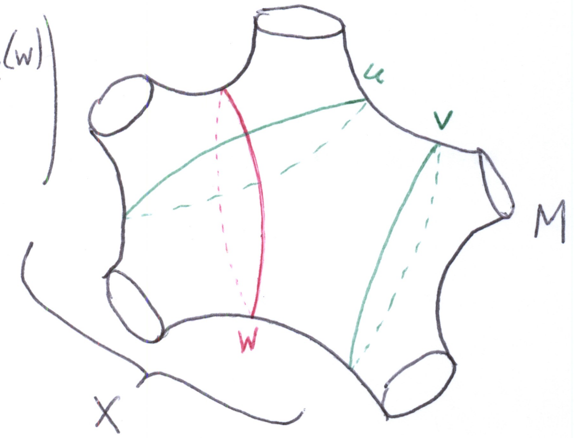
[Gromov hyperbolicity: $\mathcal{C}(M)$ is Gromov hyperbolic.

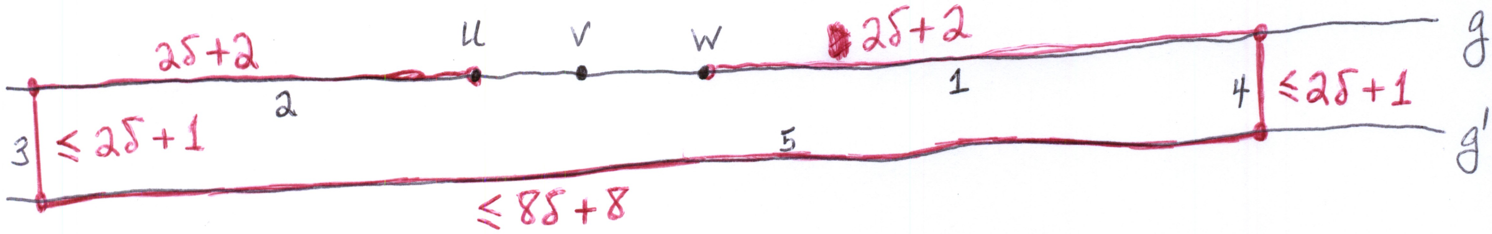
(I actually should write these in the other order.)

For now let's assume Gromov hyperbolicity with constant δ and try to get an intuition for the Bounded Geodesic Image Theorem. (This example is based on one from Masur-Minsky II.)

Suppose $M =$ . Let g be a (long) geodesic segment $\ldots, u, v, w, \ldots$. Let g' be another long geodesic segment with endpoints distance 0 or 1 from the endpoints of g. Then g & g' are $(2\delta + 1)$-fellow travelers.

(Note that $\pi_X(u)$ & $\pi_X(w)$ are distance 1 in this picture.)





Let $X \subset M$ be the closure of the 4-punctured sphere component of $M - v$. Then $u, w \subset X$.

Claim: If $\pi_X(u)$ & $\pi_X(w)$ are far apart in $\mathcal{C}(X)$ then g' must pass through v. (Not the case in the picture!)

Suppose g' does not pass through v. Form the red path in $\mathcal{C}(M)$ from u to w as shown:

1. go forward along g from w distance $2\delta + 2$
2. go backward along g from u distance $2\delta + 2$
3. skip over to g' with a path of length $\leq 2\delta + 1$ (by fellow traveler property)
4. skip over to g' with a path of length $\leq 2\delta + 1$

5. Close up with a path along g' of length $\leq 8\delta + 8$ (by the triangle inequality)

Every point of the red path is not v. So we can project it to $C(X)$.

Lemma: If $d_{C(M)}(b, c) = 1$ then $\operatorname{diam}(\pi_X(b \cup c)) \leq 2$.

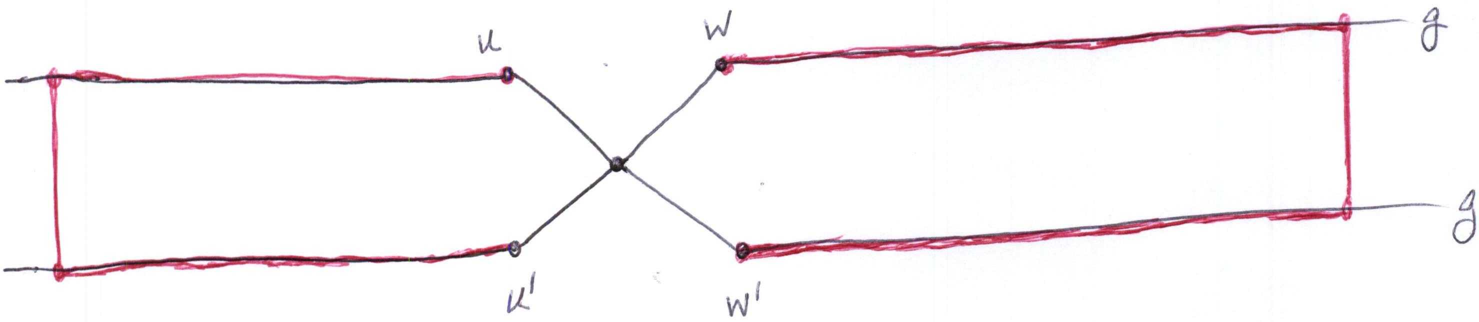
So π_X(red path) can be modified slightly to make a path of length at most $2 \cdot (2\delta + 2 + 2\delta + 2 + 2\delta + 1 + 2\delta + 1 + 8\delta + 8)$

$$= 32\delta + 28$$

We see that $d_{C(X)}(\pi_X(u), \pi_X(w)) > 32\delta + 28$

implies g' must pass through v.

Assume now that g' passes through v.



Let u' + w' be as in the picture. Using the red paths, and a similar argument, one can bound $d_{C(X)}(\pi_X(u), \pi_X(u'))$ and $d_{C(X)}(\pi_X(v), \pi_X(v'))$. It follows that geodesics $[uw]$ and $[u'w']$ are fellow travellers. All this followed from knowing that the endpoints of g + g' are close to each other.