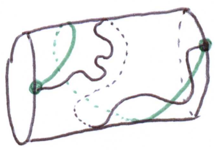
Lecture 5

Subsurface projections in the case of the torus

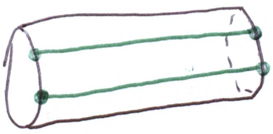
Let $\mathcal{S}$, as usual, be the set of isotopy classes of (embedded) homotopically nontrivial simple closed curves. Recall $\mathcal{S}$ is naturally identified with $P\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$. For $c \in \mathcal{S}$, $T^2 - c$ is an annulus. We need the def'n of the curve complex of an annulus, which is annoyingly complex. (No pun intended.) For the sake of culture I'll give the official def'n from Masur-Minsky's "Geometry of the complex of curves. II." For annulus A, ~~let the vertices~~ a compact annulus with boundary, let the vertices of $\mathcal{C}(A)$ be the set

$$\left\{ \begin{matrix} \text{paths from one boundary} \\ \text{component of } A \text{ to the other} \end{matrix} \right\} \Big/ \left\{ \begin{matrix} \text{homotopies } \underline{\text{fixing}} \\ \text{the boundary pointwise} \end{matrix} \right\}.$$

E.g. These are the same vertex:



These are not:



Obviously this set $\mathcal{C}(A)$ is huge, but we're stuck with it.

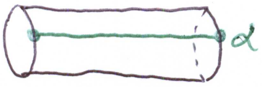
Join the vertices of $C(A)$ by an edge if they have representatives with disjoint interiors, forming the curve complex of the annulus.

For $\alpha, \beta \in C(A)$ vertices, the signed algebraic intersection number $\alpha \cdot \beta$ is well defined. (Only count interior intersections.)

Lemma: $d_{C(A)}(\alpha, \beta) = 1 + |\alpha \cdot \beta|$

Lemma: $\gamma \cdot \alpha = \gamma \cdot \beta + \beta \cdot \alpha + \varepsilon$ for $\varepsilon \in \{-1, 0, 1\}$.
(Note $\gamma \cdot \alpha = -(\alpha \cdot \gamma)$!)

Picking a base $\alpha \in C(A)$ defines a map $f : C(A) \to \mathbb{Z}$



$\beta \mapsto \beta \cdot \alpha$.

Lemma: f is a quasi-isometry, namely

$$|f(\beta) - f(\gamma)| \le d_{C(A)}(\beta, \gamma) \le |f(\beta) - f(\gamma)| + 2.$$

So without losing anything one can imagine $C(A)$ is simply $\mathbb{Z}$ (as a metric space, not a group).

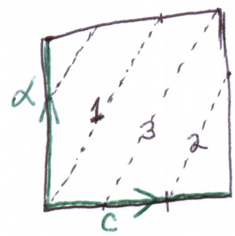
Given $c \in \mathcal{S}$ (curves on T^2) let A_c be the compact annulus obtained in the obvious way from $T - c$. For $b \in \mathcal{S} - \{c\}$ define the projection $\pi_c(b)$ to $C(A_c)$ as the set:

$$\left\{ \beta \;\middle|\; \begin{array}{l} \beta \text{ equals the restriction to } A_c \text{ of} \\ b' \text{ for some s.c.c. } b' \text{ isotopic to } b \end{array} \right\} \subset C(A_c).$$

Then $\pi_c(b)$ is a set of diameter 1.

To make this more explicit, let $c = \pm(1,0) = \frac{1}{0} = \infty \in \mathcal{S}$ and $\alpha = \pm(0,1) = 0 \in \mathcal{S}$.

Then $b = \pm(p,q) = \frac{p}{q} \in \mathcal{S}$ will project to the annulus A_c as q curves $\{b_i\}$ of slope q/p. (I guess we should have taken reciprocals somewhere.)



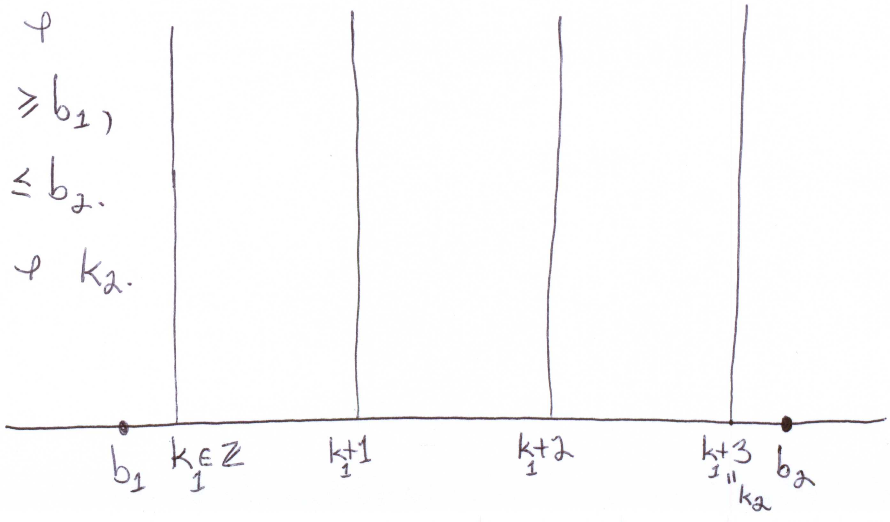
The dashed line is the $\pm(2,3) = 2/3$ curve. It has 3 components in A_c.

Each b_i will intersect α either $\lfloor p/q \rfloor$ or $\lceil p/q \rceil$ times. So the projection π_c can safely be thought of as taking the integer part of p/q.

Prop (Bounded Geodesic Image): Suppose g is a geodesic in $\mathcal{C}(T^2)$ disjoint from $c \in \mathcal{S}$. Then the diameter of the projection $\pi_c(g)$ is at most 4.

Pf: Wolog assume $c = \frac{1}{0} = \infty$. Assume $\exists$ $b_1, b_2 \in g$ s.t. the diameter of $\pi_c(b_1 \cup b_2)$ is ≥ 5. Then $b_1 = \frac{p_1}{q_1}$ & $b_2 = \frac{p_2}{q_2}$ are separated by at least 4 integer points.

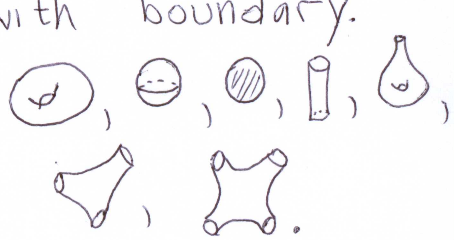
Suppose wolog $b_1 < b_2$ & k_1 is the least integer $\geq b_1$, & k_2 " " greatest " $\leq b_2$. g must pass through k_1 & k_2.



The unique geodesic from k_1 to k_2 is $\{k_1, \infty, k_2\}$. By assumption g is disjoint from ∞, yielding a contradiction. $\boxtimes$

This proposition is true in higher genus, proved by Masur-Minsky.

The gen'l curve complex

We will need to allow surfaces with boundary. Let M be a compact oriented surface, not .

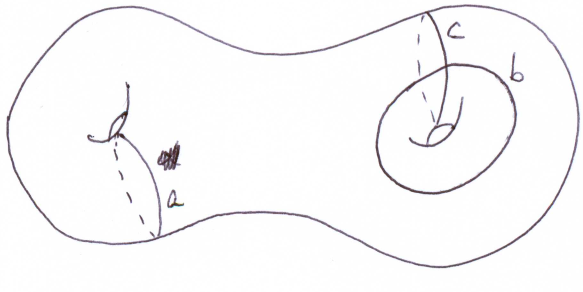
$\mathcal{S} = \left\{ \begin{array}{l} \text{isotopy classes of simple closed} \\ \text{curves not isotopic into } \partial M \end{array} \right\}$.

Then $C(M)$ is a simplicial complex with a $(k-1)$-simplex given by a pairwise disjoint k-tuple of $\mathcal{S}$. Make it a metric space by making each simplex a standard Euclidean simplex & using a path metric. We will only consider the 1-skeleton, the curve graph.

Claim: The curve graph is locally infinite.

Pf: Find $a, b, c \in \mathcal{S}$ s.t. $a \cap b = a \cap c = \emptyset$ & $b \cap c \neq \emptyset$.

(This is possible because we ruled out the "low genus" cases, & spheres with < 5 punctures.)



Then the curves $\{D_b^n(c)\}_{n\in\mathbb{Z}}$ are all distance 1 from $a \in \mathcal{S}$. $\boxtimes$

Claim: The curve graph is connected. In fact

$$d(\alpha,\beta) \leq 2\cdot i(\alpha,\beta) + 1.$$

Pf: Assume $\#(\alpha\cap\beta) = i(\alpha,\beta)$, i.e. they intersect minimally.

If $i(\alpha,\beta)=0$ then we're done.

If $i(\alpha,\beta) = 1$ then consider a small neighborhood $\mathcal{U}$ of $\alpha\cup\beta$. $\mathcal{U}$ is necessarily a punctured torus. Consider the curve $\partial\mathcal{U}$. If $\partial\mathcal{U}$ is isotopic into ∂M the M is a punctured torus. We assumed M is not a punctured torus. $\Rightarrow$ $\partial\mathcal{U} \in \mathcal{C}(M)$, $d(\partial\mathcal{U},\alpha) = d(\partial\mathcal{U},\beta) = 1$

$$\Rightarrow \quad d(\alpha,\beta) = 2.$$

Now assume $i(\alpha,\beta) = k \geq 2$ and argue by induction.

Consider a pair of adjacent points in $\alpha\cap\beta$.

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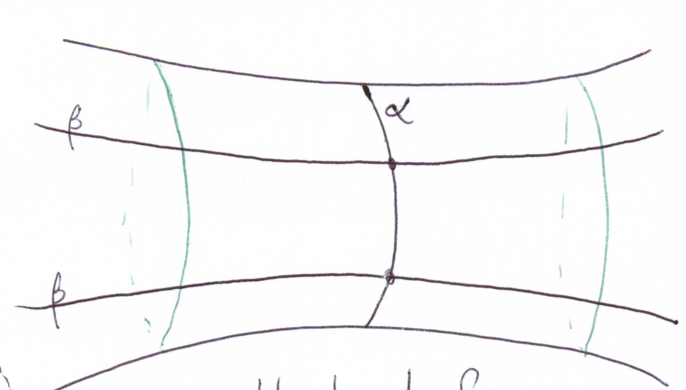
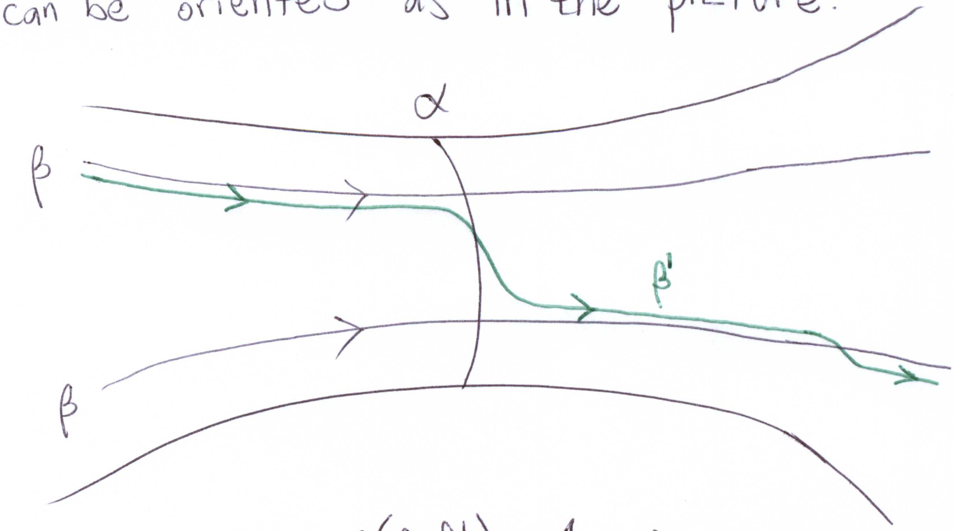


a neighborhood of α

[redacted]

Case I: Assume β can be oriented as in the picture:



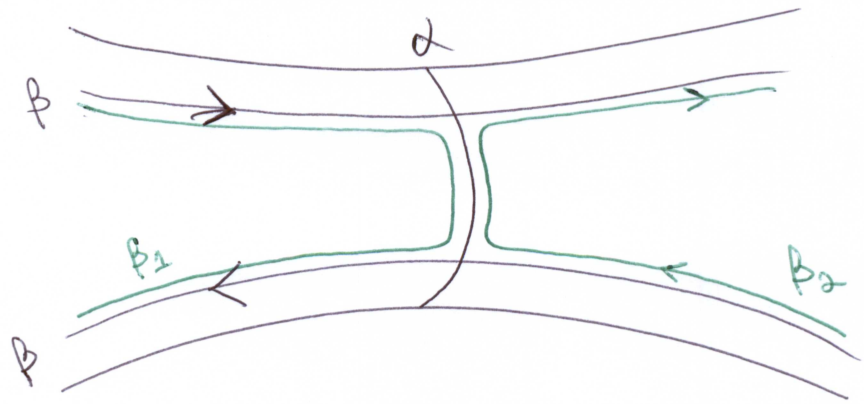
Then do a surgery as shown to produce β'. Then β' must cross β exactly once, from the left side to the right, so $i(\beta, \beta') = 1 \Rightarrow$

$d(\beta, \beta') = 2$ and β' is not isotopic into ∂M.

By induction $d(\alpha, \beta) \leq d(\alpha, \beta') + d(\beta', \beta)$

$$\leq 2(k-1) + 1 + 2 = 2k+1.$$

Case II: Assume β can be oriented as shown:



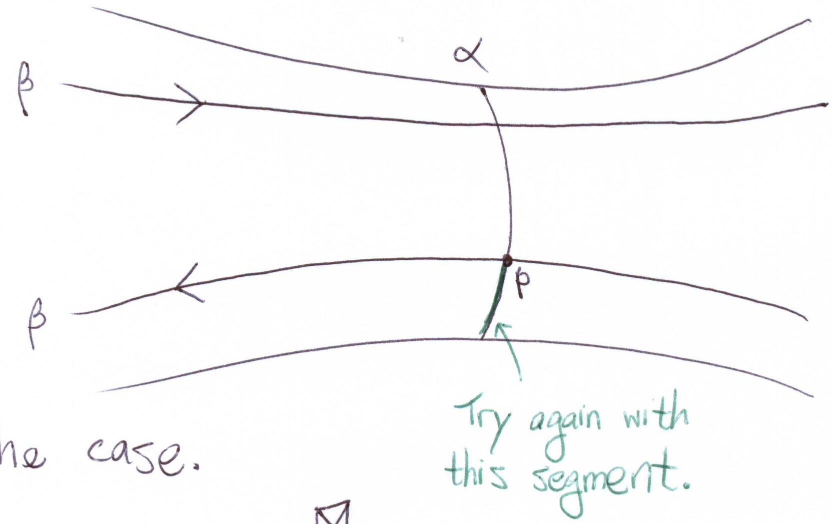
Then perform surgery to produce β_1 & β_2.

Each β_j is homot. nontrivial, & $i(\beta_j, \alpha) \leq k-2$.

However, $i(\beta, \beta_j) = 0$ so we must show at least one of the β_j is not isotopic into ∂M.

If β_j is not isotopic into ∂M then we're done by induction. Suppose β_1 & β_2 are homotopic to components of ∂M. Then the component of $M - \beta$ containing the β_j must be a thrice-punctured sphere.

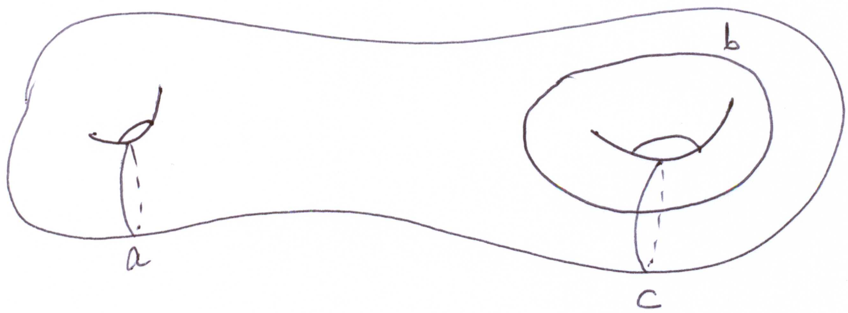
Apply the above argument to the other segment of $\{\alpha \text{ cut along } \alpha \cap \beta\}$ containing the point p as shown. If we end up again in this case then M must be a 4-times punctured sphere, which we assumed is not the case.



$\boxtimes$

Notice the is no reverse inequality; one cannot bound distance from below by intersection number.

E.g.



$$i(D_c^n(b), b) \xrightarrow{n \to \infty} \infty, \text{ but}$$

$$d(D_c^n(b), b) \leq d(D_c^n(b), a) + d(a, b) = 2.$$

So it's not obvious that the curve graph has ∞ diameter.