Lecture 4 2009-04-21

Introduction to the curve complex of the torus.

Let $\mathcal{S}$ be the set of isotopy classes of unoriented homot. nontrivial simple closed curves on the torus T^2. Recall

$$\mathcal{S} = \{\pm(0,1), \pm(1,0)\} \cup \{\pm(m,n) \mid \gcd(m,n) = 1\} \subset \frac{\mathbb{Z}\times\mathbb{Z}}{\pm 1},$$

and $i(\pm(m,n), \pm(m',n')) = \left| \det\begin{pmatrix} m & m' \\ n & n' \end{pmatrix} \right|$.

Imagine $\mathcal{S} = \mathbb{P}\mathbb{Q} = \left\{ \frac{p}{q} \in \mathbb{Q}\cup\{\infty\} \mid \gcd(p,q) = 1 \right\} \subset \partial\mathbb{H}^2 = \mathbb{R}\cup\{\infty\}$.

boundary of the upper half-plane

Turn $\mathcal{S}$ into a graph. Add an edge between $\pm(m,n)$ & $\pm(m',n')$ iff $i(\pm(m,n), \pm(m',n')) = 1$, i.e. iff $\exists$ a homeo of T^2 taking our pair to $e_1 = (1,0)$ & $e_2 = (0,1)$.

Note that any two distinct classes in $\mathcal{S}$ have <u>positive</u> intersection, so intersecting exactly once is minimal.

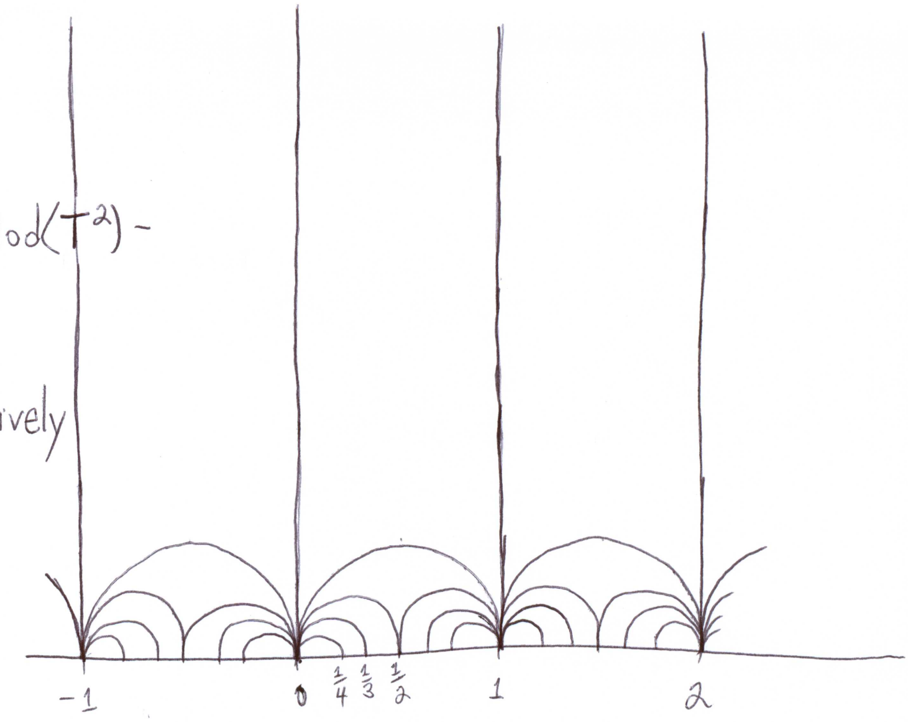
Call the resulting graph $\mathcal{C}(T^2)$, the graph of curves on T^2. $\mathcal{C}(T^2)$ is also known as the Farey graph.

Embed $\mathcal{C} = \mathcal{C}(T^2)$ into the upper half-plane $\mathbb{H}^2$ by making the edges bi-infinite geodesics between points in $\mathcal{S} = \mathbb{P}\mathbb{Q} \subset \mathbb{R}$.

The curve $\pm(1,0) = \infty$ is joined to $\begin{pmatrix} 1 & m \\ 0 & n \end{pmatrix} = \pm 1$

$\Rightarrow \pm(m,n) = \pm(m,1) = m \in \mathbb{Z}$. Similarly, $\pm(0,1) = 0$ is joined to $\pm(m,n) = \pm(1,m) = \frac{1}{m}$.

A finite piece of $\mathcal{C} \subset \overline{\mathbb{H}^2}$ looks like:



By def'n, $\mathcal{C}$ is $PSl_2\mathbb{Z} = Mod(T^2)$-invariant.

$PSl_2\mathbb{Z}$ acts transitively on $\mathbb{P}\mathbb{Q}$, implying $\mathcal{C}$ is homogeneous. (Well, at least all the vertices look the same.)

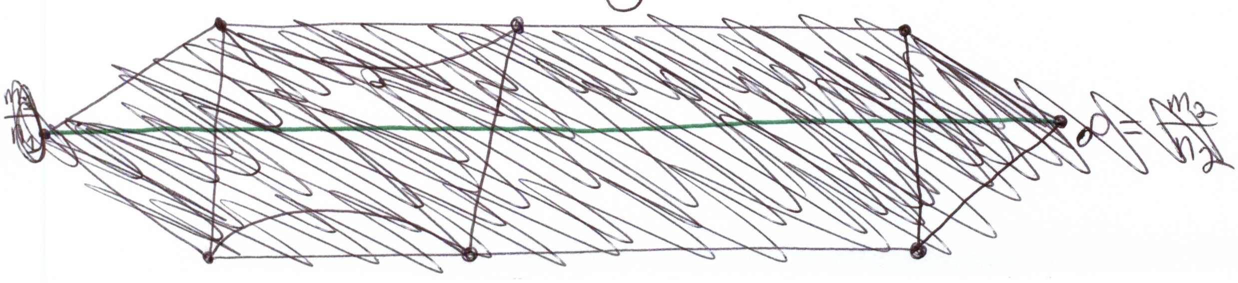
The stabilizer of $\infty = \pm(1,0)$ equals $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, which are the Dehn twists along $\infty = \pm(1,0) = e_1$.

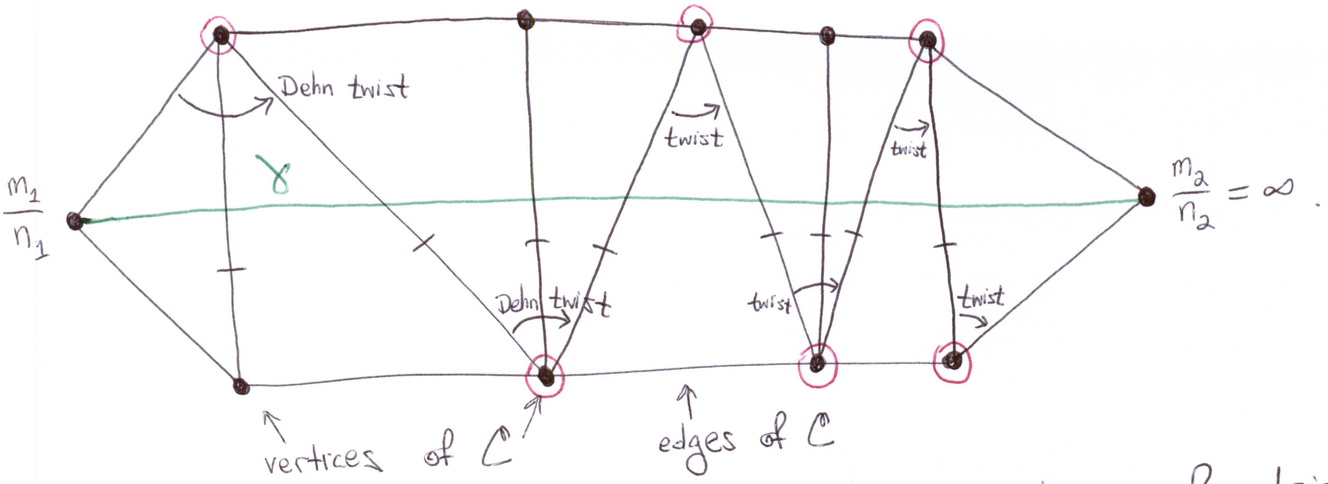
Similarly, the stabilizer of $\frac{p}{q} = \pm(p,q)$ are the Dehn twists about $\pm(p,q)$.

Note $\mathbb{C}$ is locally infinite.

Claim: $\mathbb{C}$ is connected.

Pf: Let γ be the hyperbolic geod. from $\pm(m_1,n_1) = \frac{m_1}{n_1} \in \mathbb{P}\mathbb{Q}$ to $\pm(m_2,n_2) = \frac{m_2}{n_2} \in \mathbb{P}\mathbb{Q}$. Using an elt. of $PSl_2\mathbb{Z}$ we may assume $\frac{m_2}{n_2} = \infty$. Then γ is a vertical line. $\mathbb{C}$ cuts $\mathbb{H}^2$ into ideal triangles. Combinatorially, γ hits these triangles as shown:





Once you see that only a finite number of triangles can hit γ, then connectivity follows by simply pushing γ onto the 1-skeleton of the triangles. $\boxtimes$

Claim: $\mathcal{C}$ has infinite diameter.

Pf: Each of the edges of $\mathcal{C}$ crossed by γ separate $\mathcal{C}$. So any path from $\frac{m_1}{n_1}$ to $\frac{m_2}{n_2}$ must traverse the edges marked with ticks, possibly at a vertex. ~~So the distance in $\mathcal{C}$ between $\frac{m_1}{n_1}$ & $\frac{m_2}{n_2}$ is at least [illegible]. Push this idea farther to obtain any points arbitrarily far apart, using that Dehn twists are stabilizers~~ ~~$\boxtimes$~~ Going from $\frac{m_1}{n_1}$ to $\frac{m_2}{n_2}$ in $\mathrm{Mod}(T^2)$ involves doing Dehn twists about the curves represented by the vertices circled in red, the so-called "pivots". Each pivot adds 1 to the distance. So path with many pivots must be very long. $\boxtimes$

Notice performing a large number of Dehn twists at a fixed pivot does not increase the distance.

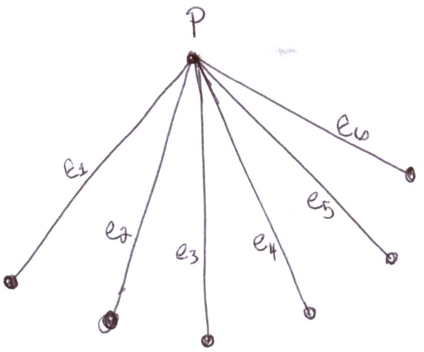
(Reference: "A geometric approach to the complex of curves on a surface" by Y. Minsky.)

Let's formalize these ideas a little.

Let v_+ and v_- be distinct points in $\mathbb{R} \cup \{\infty\}$. If $v_* \in \mathbb{Q} \cup \{\infty\}$ then think of it as an elt. of $\mathcal{S}$. ~~Join v_+ & v_- by a hyperbolic geodesic~~

Let $E(v_+, v_-)$ denote the set of edges of $\mathcal{C}$ separating v_+ & v_-. Define an order on $E(v_+, v_-)$:

$e < f \iff e$ separates v_- from the interior of f. A <u>pivot</u> is a vertex of $\mathcal{C}$ shared by ≥ 1 pair of consecutive edges of $E(v_+, v_-)$. (The pivots are circled in red in the picture on the previous page.) A <u>block</u> of pivot p is a max'l set of consecutive edges $e_1 < e_2 < \cdots < e_{w(p)}$ all sharing the vertex p. $w(p)$ is the width of the block.

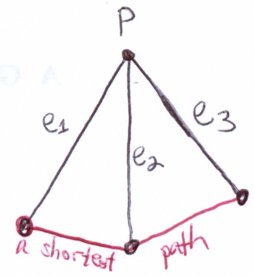


a block of width 6

~~shortest path from v_- to v_+~~

Assume for now that $v_+, v_- \in \mathcal{S}$. A shortest path from v_- to v_+ must intersect each edge of a block, possibly through a vertex. If $w(p) > 3$ then a shortest path will intersect the block in e_1 and/or

$e_{w(p)}$ only. In all cases a shortest path will first intersect the block, stay within a (closed) $\frac{1}{2}$-neighborhood of the block, and leave the block.

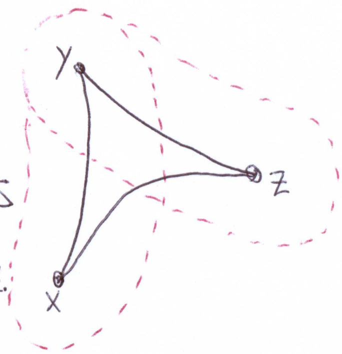


If $w(p) \leq 3$ then a shortest path may avoid p.

Prop. (Minsky?): $\mathcal{C}$ is Gromov hyperbolic.

Recall the def'n.

Def: Let X be a geodesic metric space. X is Gromov hyperbolic if $\exists$ $\delta > 0$ with the following property: Join $x, y, z \in X$ by shortest paths. Then the path from x to z is contained in the union of the δ-neighborhoods of the paths from x to y & y to z.



Pf of Proposition: It suffices to consider a triple $x, y, z \in \mathcal{S}$ & shortest paths $[xy]$, $[yz]$, & $[xz]$ in $\mathcal{C}$.

Let e be an edge of $E(x,z)$.

Either: (i) $e \in E(x,y)$ or

(ii) $e \in E(y,z)$ or

(iii) y is a vertex of e.

In case (i), $[xy]$ must hit e.

In case (ii), $[yz]$ must hit e.

In case (iii), y is in e.

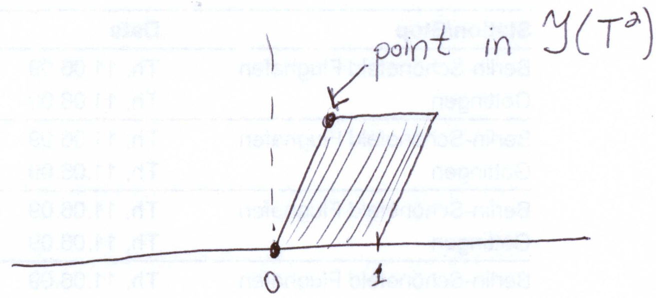
So e is inside a (closed) 1-neigh. of $[xy] \cup [yz]$. This implies the vertices of $[xz]$ are in a closed 1-neigh of $[xy] \cup [yz]$. ~~thinking about the first or last edges of [xz]~~ we see that $[xz]$ is inside a (closed) $\frac{3}{2}$-neigh. of $[xy] \cup [yz]$. $\boxtimes$

Being Gromov hyperbolic is good for many reasons. Consider a pair of infinite hyperbolic geodesics beginning at 0 and terminating at $r_1, r_2 \in \mathbb{R} - \mathbb{Q}$. These geods. determine ordered edge sequences $\{e^1_n\}$ + $\{e^2_n\}$. As soon as the sequences $e^1_1, e^1_2, e^1_3, \ldots$ + $e^2_1, e^2_2, e^2_3, \ldots$ see an unequal pair, then ~~the~~ corresponding geodesics in $\mathcal{C}$ will begin to diverge linearly. (From the model on page 4 for shortest paths we see shortest paths are unique except for finite ambiguity in traversing blocks of width ≤ 3.) From this it follows that the Gromov boundary of $\mathcal{C}$ is $\mathbb{R} - \mathbb{Q}$. Notice that $\mathcal{C} \cup \partial \mathcal{C}$ looks like $\mathbb{R} \cup \{\infty\}$, but the topology is very different. In particular, it is not compact. For example, the sequence $\{n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ has no convergent subseq.. In a sense, this is the only way a sequence can diverge: by having lots of twisting. An element of $\partial \mathcal{C} = \mathbb{R} - \mathbb{Q}$ corresponds to a measured foliation of irrational slope.

Discuss axes of hyperbolic elements of $PSL_2\mathbb{Z} = \mathrm{Mod}\,T^2$.

What's the connection between $\mathcal{C}(T^2)$ and $\mathcal{Y}(T^2)$.

Recall $\mathcal{Y}(T^2) = \{(x,y) \mid y>0\}$.



For $c \in \mathcal{C}$ let

$$U_c := \left\{ (x,y) \in \mathcal{Y}(T^2) \;\middle|\; \frac{\text{length}(c)}{\text{area}} \leq \varepsilon \right\}.$$

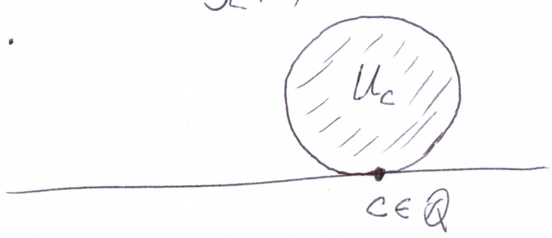
$= \begin{pmatrix} \text{flat tori where } c \\ \text{is not long} \end{pmatrix}$

ε won't be very small, just a medium size

For example $$U_\infty = \left\{ (x,y) \;\middle|\; \frac{\text{length}(\infty)}{\text{area}} = \frac{\text{length}(\pm(1,0))}{\text{area}} = \frac{1}{y} \leq \varepsilon \right\}$$

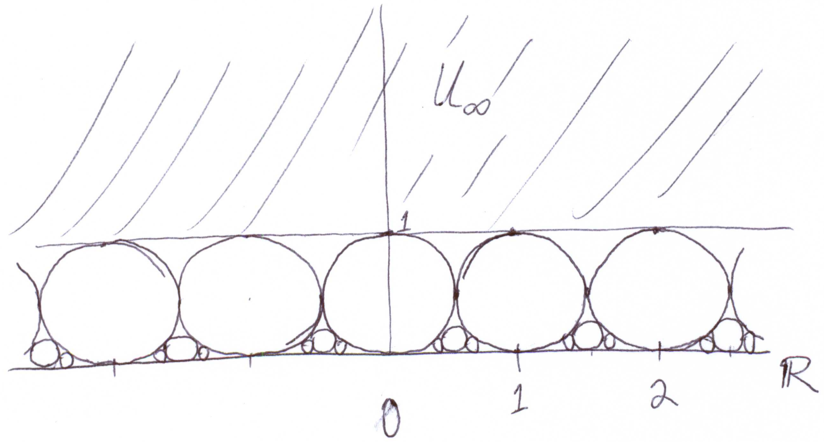
$$= \left\{ (x,y) \;\middle|\; y \geq \frac{1}{\varepsilon} \right\}.$$

Apply $\mathrm{Mod}(T^2) = PSL_2\mathbb{Z}$ to U_∞ and conclude that U_c for $c \neq \infty$ is a horodisk tangent to c.



FACT: $\sup_{\varepsilon} \{ U_c \text{ are pairwise disjoint} \} = 1$.

If $\varepsilon = 1$ then it looks something like



Consider the nerve of the collection of closed sets $\{U_c\}_{c\in\mathcal{S}}$.
This nerve is a graph with vertices U_b & U_c joined by an edge iff $U_b \cap U_c \neq \emptyset$. (The higher skeleta are empty.)
This nerve is exactly $\mathcal{C}(T^2)$. This is not an accident. The collection $\{U_c\}_{c\in\mathcal{S}}$ is often called the thin parts of Teichmüller space. Then $\mathcal{C}$ is the nerve of the thin parts. This will persist in higher genera.