Lecture 1 at HUJI 2009-03-18

(Notes from a course on mapping class groups by Peter Storm at Hebrew University.)

Fix a compact oriented surface M without boundary. Assume all maps preserve orientation unless explicitly stated otherwise. Assume M is not a sphere.

Def: $\begin{pmatrix}\text{Mapping class} \\ \text{group of } M\end{pmatrix} = \mathrm{Mod}(M) := \pi_0(\mathrm{Homeo}(M))$ (orientation preserving homeos)

We will call elements of $\mathrm{Mod}(M)$ mapping classes.

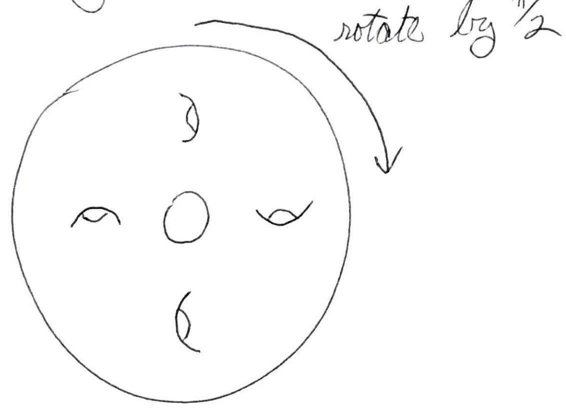
Thm (Dehn - Nielsen - Baer): The natural map

$$\mathrm{Homeo}(M) \longrightarrow \mathrm{Out}(\pi_1 M)$$ defines a homom.

$\mathrm{Mod}(M) \to \mathrm{Out}(\pi_1 M)$. This homom is injective with image of index 2. The image is exactly orientation preserving elts of $\mathrm{Out}(\pi_1 M)$, i.e. those acting trivially on $H^2(\pi_1 M; \mathbb{Z}) \cong \mathbb{Z}$.

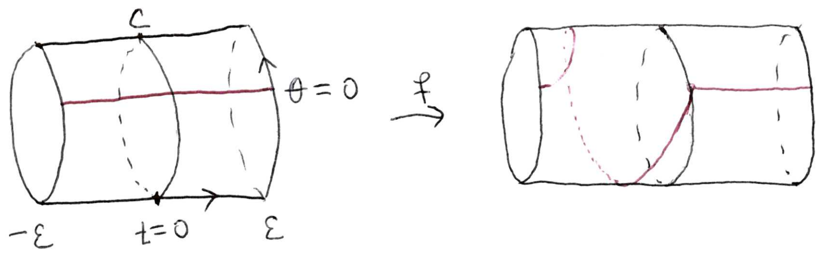
E.g. if M is a torus then $\mathrm{Mod}(M) \cong Sl_2\mathbb{Z}$.

An example of a mapping class is the finite order homeom:



Another (more important) example is given by the Dehn twist. Let $c \subset M$ be an (embedded) homotopically nontrivial simple closed curve. A right hand Dehn twist about c, D_c, is a homeom $M \to M$ with support in a small annular neigh. of c.

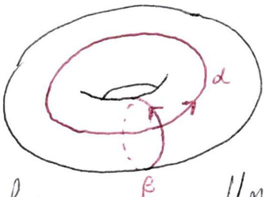
Put "coordinates" (t,θ) on an annular neigh. of c s.t. $t \in (-\varepsilon, \varepsilon)$, ~~and~~ $\theta \in S^1$, and $c = \{(0,\theta) \mid \theta \in S^1\}$.



Then define a homeo $f: M \to M$ with support in this annular nbd by

$$f(t,\theta) := \begin{cases} (t, e^{\frac{t}{\varepsilon}\cdot 2\pi i}) & \text{for } t \le 0 \\ (t,\theta) & \text{for } t > 0 \end{cases}$$

Then D_c is the mapping class of f. D_c is also called the left ~~handed~~ Dehn twist about c. (I chose the left-handed orientation to follow N. Ivanov.) Note the def'n of f involved an orientation of c, namely which is the left side of c, but D_c is independent of this choice. Obviously, we could have chosen f to be smooth with slightly more work.

On a torus 

, Dehn twists take a particularly simple linear form. Under the Dehn-Nielsen-Baer ~~morph~~ homom. D_β corresponds to

$$\left\{\begin{array}{l} \beta \longmapsto \beta \\ \alpha \longmapsto \alpha + \beta \end{array}\right\} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and D_α corresponds to

$$\left\{\begin{array}{l} \alpha \longmapsto \alpha \\ \beta \longmapsto \beta - \alpha \end{array}\right\} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Here Dehn twists correspond to unipotent elements. Intuitively, Dehn twists are similar to unipotents in higher genera.

In lattices, unipotents are not "typical" elements. This is true here also. Without defining "typical" precisely, I claim Dehn twists are not typical elements of Mod(M). What is an example of a typical element? Consider $\varphi = D_\alpha^{-1} D_\beta$, which corresponds to $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, a hyperbolic element of $SL_2\mathbb{Z}$. I claim φ is typical. In particular, φ has the following property:

[Def: Let $\mathcal{S}$ be ~~a~~ the set of isotopy classes of homotopically nontrivial unoriented simple closed curves on M.

Clearly $\mathrm{Mod}(M) \curvearrowright \mathcal{S}$.

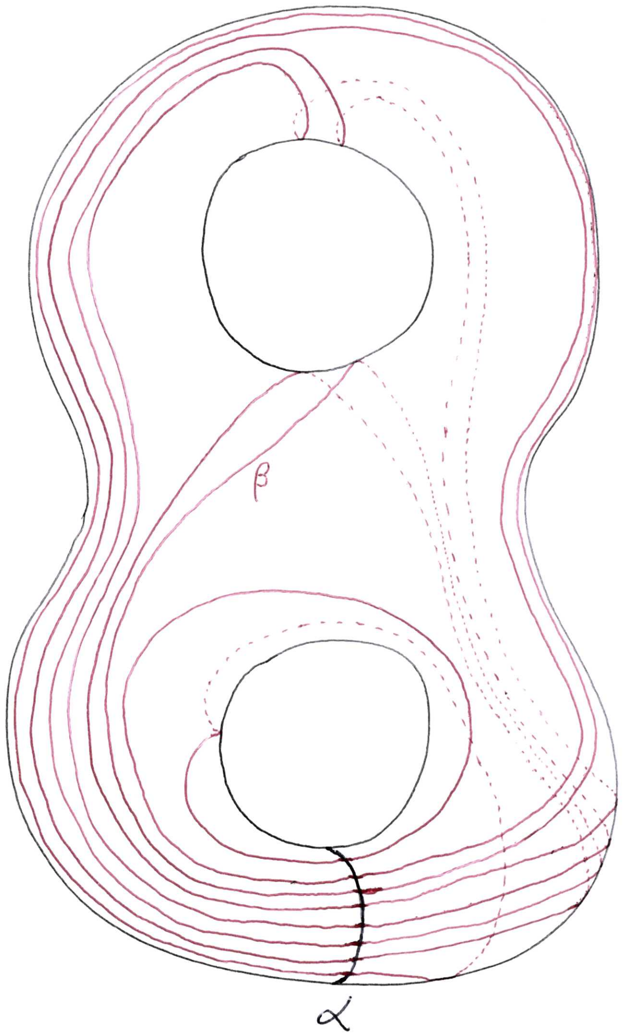
(✱) [Then for all $k \neq 0$, φ^k acts on $\mathcal{S}$ without fixed pts., i.e. if c is a nontrivial s.c.c. then $\varphi^k(c)$ is not homotopic to c.]

We'll call this property (✱). In this case it is easy to prove using linear algebra. For , $\mathcal{S}$ is $\left\{ \pm(m,n) \;\middle|\; \begin{array}{l} \text{If } m=0 \text{ then } n\neq 0. \\ \text{If } n=0 \text{ then } m \neq 0. \\ \text{Otherwise } \gcd(m,n)=1 \end{array} \right\}$.

How can be build higher genus mapping classes with property (✱)? Suppose α and β are simple closed curves on M satisfying

- they are homotopically nontrivial
- $M - (\alpha \cup \beta)$ is a set of disks
- If $\alpha' \sim \alpha$ and $\beta' \sim \beta$ then $\#|\alpha \cap \beta| \leq \#|\alpha' \cap \beta'|$, i.e. α & β intersect minimally.

For example if M has genus 2 then one could choose α & β as:



Then I claim that $D_\alpha^{-1} D_\beta$ has property (★) and should be considered a typical element of $\mathrm{Mod}(M)$. See (FLP, Ex. 13).

Measured singular foliations

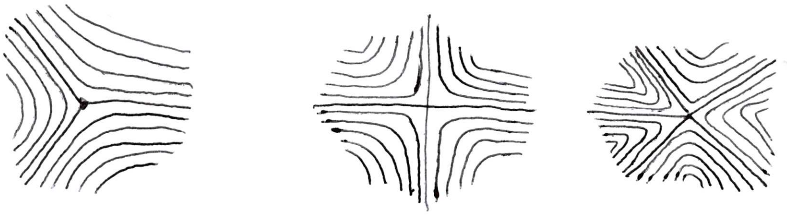
A measured singular foliation $\mathcal{F}$ is:

- a finite set $S = \{p_1, p_2, \dots, p_N\} < M$ (the "singular set")
- an atlas $\{U_\alpha, \varphi_\alpha\}$ on $M - S$ such that each $U_\alpha = (a_1, b_1) \times (a_2, b_2)$, each transition $\varphi_\beta^{-1} \circ \varphi_\alpha$ takes horizontal lines to horizontal lines, and each transition $\varphi_\beta^{-1} \circ \varphi_\alpha$ preserves vertical distances. More specifically, these conditions say:

$$\left(\begin{array}{l} \text{If } (\varphi_\beta^{-1} \circ \varphi_\alpha)(x_i, y_i) = (x_i', y_i') \text{ for } i \in \{1,2\} \text{ then} \\ \qquad |y_1 - y_2| = |y_1' - y_2'|. \end{array} \right)$$

This atlas defines a horizontal foliation on $M - S$.

- Each singular point p_i has a neigh. such that the horizontal foliation looks like one of

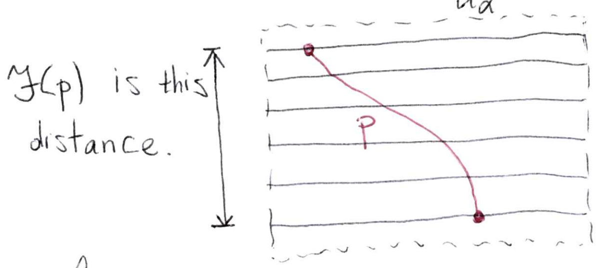


, ... (allow higher order singularities)

$\mathcal{F}$ defines a (positive) measure on the set of arcs of $M-S$ transverse to the horizontal foliation. Namely, if such an arc $p:[0,1]\to M-S$ is contained in a single chart then

$$\mathcal{F}(p) = \left| (y\text{-coord. of } p(1)) - (y\text{-coord of } p(0)) \right|.$$

In gen'l, cut the arc into small pieces and sum.



Note that if $\chi(M)<0$ then the singular set S must be nonempty.

We will say two foliations $\mathcal{F}$ & $\mathcal{F}'$ are isotopic if there is a diffeo f on M isotopic to the identity such that:

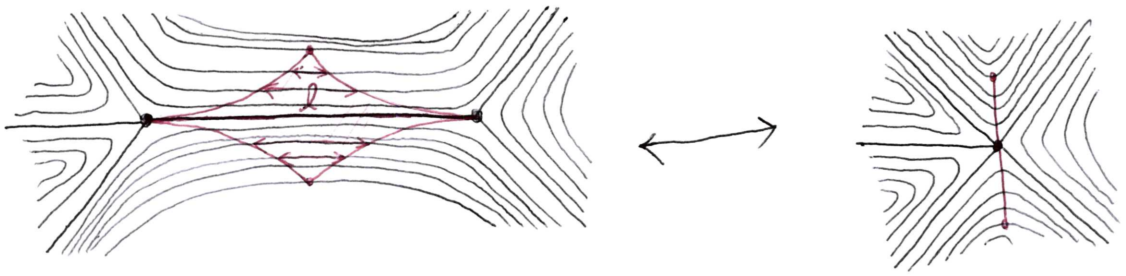
- $f(S) = f(S')$ ← singular set of $\mathcal{F}'$

- f sends the horiz. foliation of $\mathcal{F}$ to that of $\mathcal{F}'$.

- If p is an arc in $M-S$ transverse to $\mathcal{F}$ then
$$\mathcal{F}'(f(p)) = \mathcal{F}(p).$$

E.g. One could compose the charts with a small diffeo of M.

Next we'll describe Whitehead moves. Suppose l is a leaf of $\mathcal{F}$ running from one singular pt to another as shown.



In a nbd. of l cut out an open diamond shaped region as shown, and glue the endpts of horiz. leaves together. This produces the measured singular foliation on the right. This process is reversible. Both are called Whitehead moves. The two measured singular foliations are Whitehead equivalent.

Let A be the set of measured singular foliations on M.

Define

$$\mathcal{F}_1 \sim_1 \mathcal{F}_2 \iff \left\{ \begin{array}{l} \exists \text{ a sequence of Whitehead} \\ \text{moves on } \mathcal{F}_1 \text{ producing a measured} \\ \text{sing. foliation isotopic to } \mathcal{F}_2 \end{array} \right\}$$

and

$$\mathcal{F}_1 \sim_2 \mathcal{F}_2 \iff \left\{ \begin{array}{l} \exists\ \mathcal{F}_3 \text{ s.t. } \mathcal{F}_1 \sim_1 \mathcal{F}_3 \text{ and the} \\ \text{measure on } \mathcal{F}_3 \text{ is a constant multiple} \\ \text{of the measure on } \mathcal{F}_2 \end{array} \right\}.$$

Then $\mathcal{MF} := A/\sim_1$ and $\mathbb{PMF} := A/\sim_2$.

Def: $\mathcal{F}^u$ & $\mathcal{F}^s$ foliations on M are transverse if they have the same singular set S and are transverse on $M - S$.

Def: $f: M \to M$ a homeomorphism is pseudo-Anosov (Ψ-A) if $\exists$ $\mathcal{F}^u$ & $\mathcal{F}^s$ transverse singular measured foliations and $\lambda > 1$ s.t.

- f sends leaves of $\mathcal{F}^*$ to leaves of $\mathcal{F}^*$
- $f(\mathcal{F}^s) = \frac{1}{\lambda} \mathcal{F}^s$ and $f(\mathcal{F}^u) = \lambda \mathcal{F}^u$,

  where these are equalities of measures.

Def: A multicurve $c \subset M$ is an embedded 1-manifold (i.e. a finite union of simple closed curves) s.t. each component is homotopically nontrivial, and no pair of components are isotopic.

Thurston's classification of surface homeos: A homeom $f: M \to M$ is isotopic to a homeo g satisfying at least one of the following:

(i) g has finite order in Homeo(M)

(ii) $\exists$ multicurve $c \subset M$ s.t. $g(c) = c$.

(iii) g is pseudo-Anosov.

(If (iii) then not (i) and not (ii).)

Lecture 2 at HUJI 2009-03-25

Recall the def'n of Teichmüller space. Fix a closed oriented surface M of genus $g > 1$. (We'll draw M with genus 2.) Define the set of pairs $\{(X,m)\}$ where X is a hyperbolic surface and $m: M \to X$ is a ~~homeomorphism~~ homeomorphism. ("m" stands for "marking".) Define the equivalence relation $\sim$:

$$(X,m_X) \sim (Y,m_Y) \iff \left(\begin{array}{l} \exists \text{ isometry } f: X \to Y \\ \text{s.t. } f \circ m_X \text{ is homotopic to } m_Y. \end{array} \right)$$

~~Can replace homotopic with isotopic without changing $\sim$.~~

Then $\mathcal{T}(M) := \{(X,m)\} / \sim$. So far, this is only a set. Define a metric

$$d((X,m_X),(Y,m_Y)) := \inf \left\{ \log K \;\middle|\; \begin{array}{l} \exists \text{ K-bilipschitz homeom. } f: X \to Y \\ \text{such that } m_Y \underset{\text{homotopic}}{\sim} f \circ m_X \end{array} \right\}$$

on $\mathcal{T}(M)$, turning it into a metric space. (This metric is not particularly interesting, but it easy to define. All the well-known metrics on $\mathcal{T}(M)$ define the same topology.)

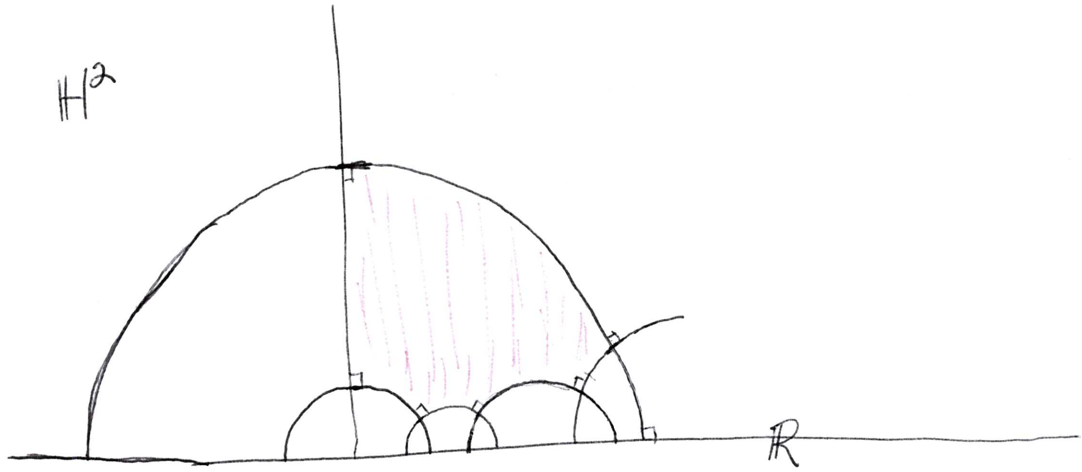
FACT: $\mathcal{T}(M) \overset{\text{homeo.}}{=} \mathbb{R}^{6g-6}$ (due to maybe Teichmüller or Bers?)

$\mathrm{Mod}(M) \curvearrowright \mathcal{T}(M)$ by isometries via:

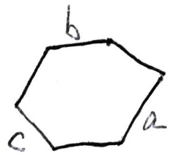
$$\varphi \cdot (X, m_X) := (X, m_X \circ \varphi^{-1}).$$

Note this is a mapping class while this is a homeo.
Check this is well defined!

How can we build a hyperbolic surface? Begin with some planar (hyperbolic) geometry. $\exists$ right-angled hexagons in $\mathbb{H}^2$, e.g.

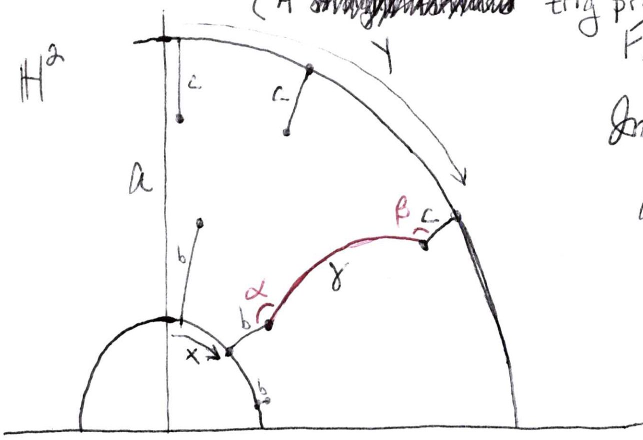


Prop: Label the edges of a hexagon as shown:

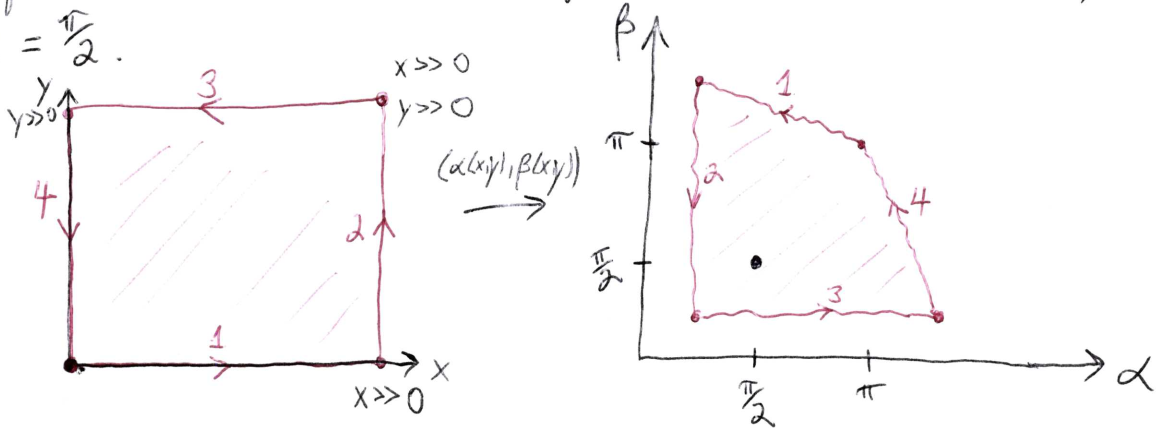


Pick $l_a, l_b, l_c > 0$. $\exists!$ right-angled hyper. metric on the hexagon such that edge $*$ has length l_* (for $* \in \{a, b, c\}$).

Pf sketch: (This proof is "Thurston-esque".) Wolog $a > b > c$.
(A trig proof is possible. See Ratcliffe Thm 3.5.14.)



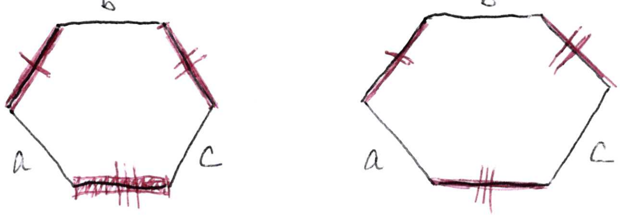
Fix a as a vertical geodesic in $\mathbb{H}^2$. Imagine b & c swinging along geodesics as shown. Let x be the distance from a to b. " y " " " " a " c. Given x & y, let γ be the geod. from b to c. Label the angles of γ as shown. The goal is to find x & y s.t. $\alpha = \beta = \frac{\pi}{2}$.



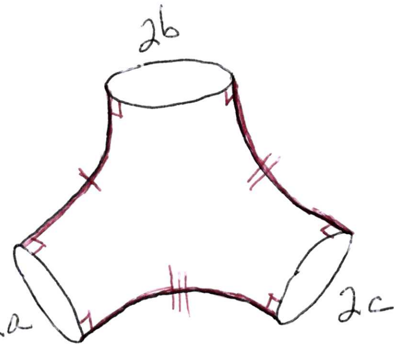
Examine the behavior of α & β when at least one of x, y is very large. This is shown in the picture. By continuity $\exists$ values of x & y producing $\alpha = \beta = \pi/2$. $\boxtimes$

(Note: This proof sketch does not discuss uniqueness.)

Given two isometric right-angled hyper. hexagons,



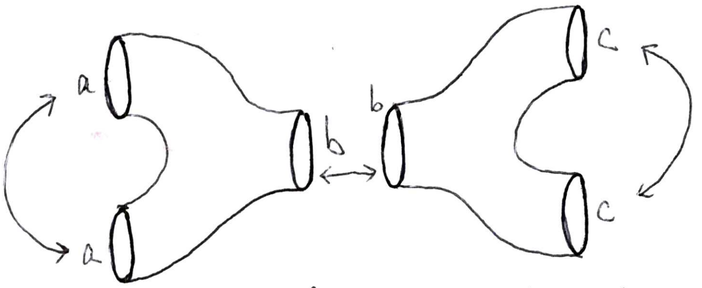
glue along the red edges to produce hyperbolic pants.

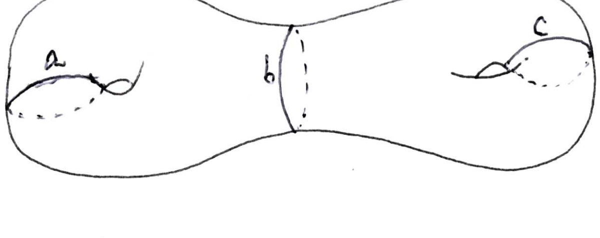


Similarly, given hyperbolic pants with geodesic boundary, add the red geodesics and cut to obtain right-angled hyperbolic hexagons.

Cor: $\forall\ l_1, l_2, l_3$ $\exists!$ (marked) pants with ∂ curves of length $l_1, l_2,$ & l_3.

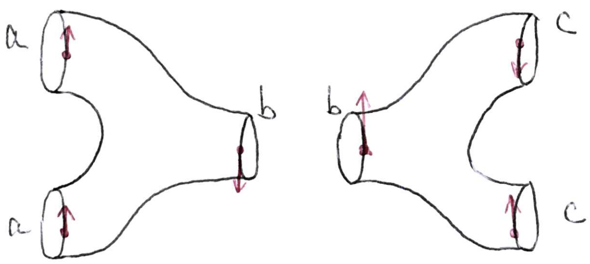
Given two pants with boundary lengths as shown, we can glue to obtain a genus 2 hyper. surface. Similar constructions work in higher genus.

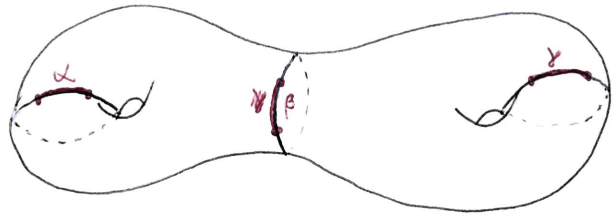




Note there is ambiguity in the gluing.

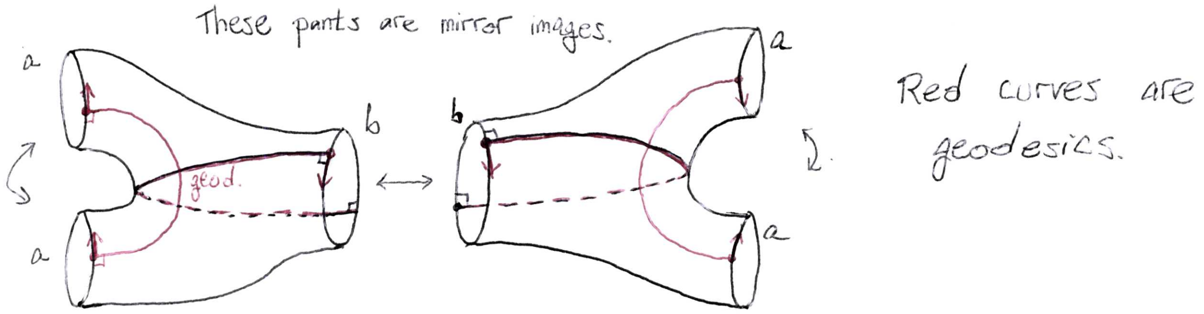
To explicate the ambiguity, add an oriented point to each boundary circle. Then a, b, & c in the closed surface each

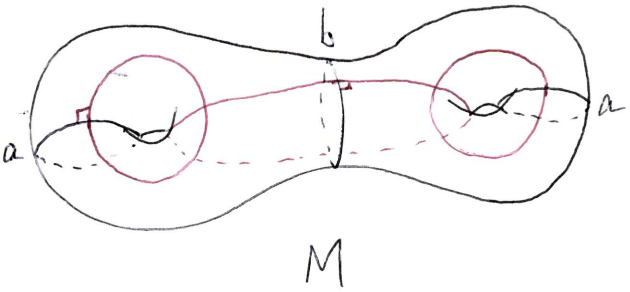




have a well-defined twist $\alpha, \beta, \gamma \in [0, 2\pi)$. Intuitively this indicates that we must specify 6 real parameters $a, b, c, \alpha, \beta, \gamma$ to build a hyperbolic ~~space~~ genus two surface, suggesting the Teichmüller space $\mathcal{T}$(genus two surface) should have dim'n 6. This is correct, but not a proof. A similar construction in higher genera shows that $\mathcal{T}(M)$ should have dim'n $6g-6$, and this count is correct.

Next I'll describe the Fenchel-Nielsen coord. system on $\mathcal{T}(M)$. For simplicity I'll describe it when M has genus 2. Let's put the following nice hyper. metric on M.





M

As before, for any l_1, l_2, l_3 and $\theta_1, \theta_2, \theta_3$ near 0 I can build a hyperbolic genus two surface $X(l_1, l_2, l_3, \theta_1, \theta_2, \theta_3)$ where l_1 is the length of the left curve,

l_2 " " " " " middle " ,

l_3 " " " " " right " ,

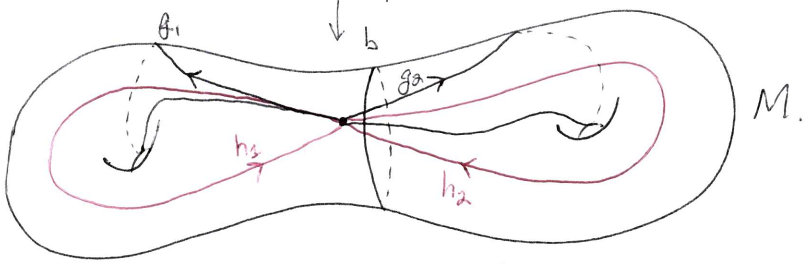
θ_1 is the twist of the left curve,

θ_2 " " " " " middle " , &

θ_3 " " " " " right curve.

Specify a marking $m: M \to X(\vec{l}, \vec{\theta})$ by defining generators for $\pi_1 M$:

The basepoint is to the left of b.



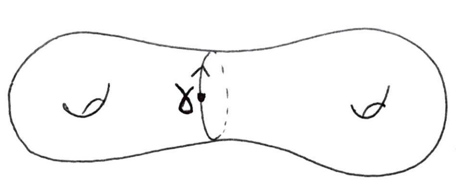
For small θ_i, mark $X(\vec{l}, \vec{\theta})$ by a homeom. taking g_i (on M) to g_i (on X) & h_i (on M) to h_i (on X). How to extend the for large θ_i? We adjust the marking. Specifically, if $\theta_i \in \mathbb{R}$ build a hyperbolic surface $X(\vec{l}, \vec{\theta})$ using twist parameters (θ_i (mod 2π)) and define a marking taking

$$(g_1 \text{ on } M) \longmapsto (g_1 \text{ on } X(\vec{l}, \vec{\theta}))$$

$$(h_1 \text{ on } M) \longmapsto \left(g_1^k h_1 \text{ on } X(\vec{l}, \vec{\theta}) \text{ where } k \text{ is the integer floor of } \theta_1/2\pi, \text{ i.e. the greatest int.} \leq \theta_1/2\pi, \text{ i.e. } \lfloor \theta_1/2\pi \rfloor \right)$$

$$(g_2 \text{ on } M) \longmapsto \left(\gamma^{-k_2} g_2 \gamma^{k_2} \text{ on } X(\vec{l},\vec{\theta}) \text{ where } k_2 = \lfloor \theta_2/2\pi \rfloor\right)$$

$$(h_2 \text{ on } M) \longmapsto \left(\gamma^{-k_2} g_2^{-k_3} h_2 \gamma^{k_2} \text{ on } X(\vec{l},\vec{\theta}) \text{ where } k_3 = \lfloor \theta_3/2\pi \rfloor\right)$$

and $\gamma = g_1 h_1^{-1} g_1^{-1} h_1$ is the curve 

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Abusing notation slightly, let g_i also denote the s.c.c. in the free homotopy class of g_i. Then the marking is better described as a ~~the homotopy equivalence~~ homeomorphism with the following action on isotopy classes of scc's.

$$\left.\begin{aligned}
(g_1 \text{ on } M) &\longmapsto (g_1 \text{ on } X(\vec{l},\vec{\theta})) \\
(h_1 \text{ on } M) &\longmapsto (D_{g_1}^{k_1} h_1 \text{ for } k_1 = \lfloor \theta_1/2\pi \rfloor) \\
(g_2 \text{ on } M) &\longmapsto (D_{\gamma}^{+k_2} h_1 \text{ for } k_2 = \lfloor \theta_2/2\pi \rfloor) \\
(h_2 \text{ on } M) &\longmapsto (D_{\gamma}^{k_2} D_{g_2}^{k_3} h_2 \text{ for } k_3 = \lfloor \theta_3/2\pi \rfloor)
\end{aligned}\right\} (\dagger)$$

Recall D_* is always a right-hand twist along $*$, regardless of any orientation $*$ may have.

This defines a marking $m_{X(\vec{l},\vec{\theta})}$: just apply Dehn twists as prescribed in $(\dagger)$ to the original identity marking

$$m_{X(a,b,a,0,0,0)} : M \longrightarrow X(a,b,a,0,0,0).$$

So for all $l_1, l_2, l_3 > 0$ & $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ we have defined a point $(X(\vec{l},\vec{\theta}), m_{X(\vec{l},\vec{\theta})})$ in $\mathcal{T}(M)$.

This defines a set map

$$FN: (0,\infty)^3 \times \mathbb{R}^3 \longrightarrow \mathcal{T}(M)$$

Thm (Fenchel-Nielsen): This map is a homeomorphism.

More generally, in higher genus we can define

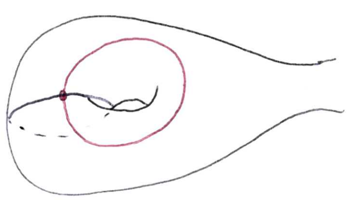
$$FN: (0,\infty)^{3g-3} \times \mathbb{R}^{3g-3} \longrightarrow \underset{\substack{\uparrow \\ \text{genus } g}}{\mathcal{T}(M)},$$

& this map is always a homeom..

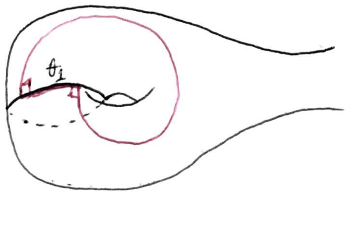
(In fact, for any $\varphi \in \mathrm{Mod}(M)$, $(FN^{-1} \circ \varphi \circ FN)$ is a real-analytic diffeom..
We won't pursue this further.)

This gives an explicit mental picture of $\mathcal{T}(M)$ in terms on "length-twist" coordinates. Let's see an example using the above notation. We look at the left side of our surface as θ_1 increases past 2π. Fix l_1, l_2, l_3, & $\theta_2 = \theta_3 = 0$.

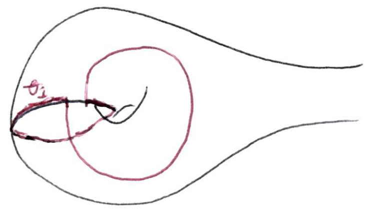
$\theta_1 = 0$



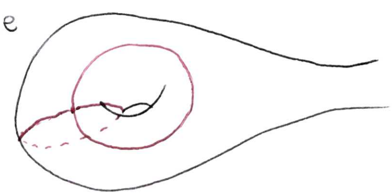
$0 < \theta_1 < 2\pi$



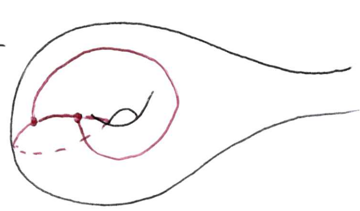
$\theta_1 < 2\pi$ but near 2π



$\theta_1 = 2\pi$, change the marking by D_{g_1}.



$4\pi > \theta_1 > 2\pi$, the marking is again D_{g_1} times the original



This completes our description of Fenchel-Nielsen coordinates.

Define
$$\mathcal{S} = \left\{ \begin{array}{l} \text{isotopy classes of homotopically} \\ \text{nontrivial unoriented} \\ \text{simple closed curves on } M \end{array} \right\}.$$

For $c \in \mathcal{S}$ define
$$\begin{aligned} \ell_c : \mathcal{T}(M) &\longrightarrow (0,\infty) \\ (X,m) &\longmapsto \inf \left\{ \text{length}(c') \;\middle|\; \begin{array}{l} c' \subset X \text{ homotopic} \\ \text{to } m(c) \end{array} \right\} \end{aligned}$$

FACT: $\ell_c((X,m))$ is always realized by the length of a simple closed geodesic $c' \subset X$ homotopic to $m(c)$.

Let $(0,\infty)^{\mathcal{S}}$ denote the space of maps $\mathcal{S} \to (0,\infty)$ with the topology of pointwise convergence (aka the product topology).
The we have

$$\begin{aligned} \ell_* : \mathcal{T}(M) &\longrightarrow (0,\infty)^{\mathcal{S}} \\ (X,m) &\longmapsto \{ c \mapsto \ell_c(X,m) \}. \end{aligned}$$

Thm (Thurston): ℓ_* is a homeomorphism onto its image.

This homeom. is proper.

More is true. Let $\pi : (0,\infty)^{\mathcal{S}} \longrightarrow \mathbb{P}(0,\infty)^{\mathcal{S}}$ denote projectivization.

Thm (Thurston): $\pi \circ \ell_*$ is a homeom. onto its image.

Recall the def'n of a measured singular foliation, $\mathcal{F}$, on M.

For a s.c.c. $\alpha \subset M$ define

$$\int_\alpha \mathcal{F} = \sup \left\{ \sum \text{measure}(\alpha_i) \;\middle|\; \begin{array}{l} \alpha_1, \dots, \alpha_\ell \text{ disjoint open subarcs of } \alpha \\ \text{transverse to } \mathcal{F} \end{array} \right\}$$

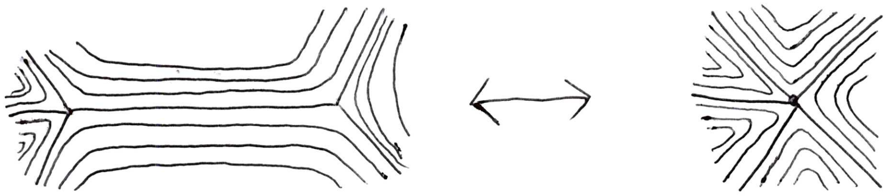
$= \left(\text{total variation on the measure of } \mathcal{F} \text{ restricted to } \alpha\right).$

and for $c \in \mathcal{S}$ define $I(\mathcal{F}, c) = \inf\limits_{\alpha \sim c} \int_\alpha \mathcal{F}$.

I stands for intersection. It's possible for $I(\mathcal{F}, c) = 0$, e.g. if c is a closed leaf of $\mathcal{F}$.

Recall $\mathcal{MF}$ is the <u>set</u> of measured singular foliations modulo 2 equivalences:

- isotopy
- Whitehead equivalence



Claim: $I : \mathcal{MF} \times \mathcal{S} \longrightarrow [0, \infty)$ is well-defined.

Taken together, these define a map

$$I_* : \mathcal{MF} \longrightarrow [0, \infty)^{\mathcal{S}}$$

Thm (Thurston): I_* is injective with image disjoint from $\vec{0}$.

Use I_* to define a topology on $\mathcal{MF}$. Let $\mathbb{P}\mathcal{MF}$ denote projective classes of ~~measured singular~~ $\mathcal{MF}$.

Thm (Thurston): $\pi \circ I_* : \mathcal{MF} \to \mathbb{P}[0,\infty)^{\mathcal{S}}$ induces a map

$$\mathbb{P}\mathcal{MF} \xrightarrow{I_*} \mathbb{P}[0,\infty)^{\mathcal{S}}$$ that is a homeom. onto its image.

Moreover, $I_*(\mathbb{P}\mathcal{MF}) \overset{\text{homeo}}{=} S^{6g-7}$.

Thm (Thurston): Consider $\pi \circ \ell_*(\mathcal{Y})$, $I_*(\mathbb{P}\mathcal{MF}) \subset \mathbb{P}[0,\infty)^{\mathcal{S}}$.

- $\overline{\pi \circ \ell_*(\mathcal{Y})} = I_*(\mathbb{P}\mathcal{MF})$
- $(\pi \circ \ell_*(\mathcal{Y})) \cup (I_*(\mathbb{P}\mathcal{MF}))$ is homeom. to a closed ball.
- Mod(M) acts on this closed ball by homeomorphisms.

Recall def'ns of $\mathcal{Y}(M)$, $\mathcal{S}$, ℓ_*, I_*, $\pi : [0,\infty)^{\mathcal{S}} \to \mathbb{P}([0,\infty)^{\mathcal{S}})$.

Mod(M) acts on $\mathcal{S}$ in the obvious way: $\varphi \cdot c := \varphi(c)$.

Then $\mathrm{Mod}(M) \curvearrowright [0,\infty)^{\mathcal{S}}$ as: $(\varphi \cdot f)(c) = f(\varphi^{-1} \cdot c)$.

With this, ℓ_* and I_* are equivariant. Let's check ℓ_*.

$$\ell_*(\varphi \cdot (X,m)) = \ell_*((X, m \circ \varphi^{-1})) = \{ c \longmapsto \ell_c(X, m \circ \varphi^{-1}) \}$$

$$= \{ c \longmapsto \ell_{\varphi^{-1}(c)}(X,m) \} = \varphi \cdot \ell_*((X,m)).$$

Recall ℓ_* is an embedding. This means a surface is determined by the lengths of its scc's. $\pi \circ \ell_*$ is also an embedding, meaning it's impossible in hyperbolic geometry to simultaneously expand all the scc's by the same factor.

To better understand I_*, let's begin with intersections of scc's.

$$i : \mathcal{S} \times \mathcal{S} \longrightarrow [0,\infty)$$
$$(b, c) \longmapsto i(b,c)$$

is the $\min\limits_{\substack{b' \sim b \\ c' \sim c}} \# | b' \cap c' |$.

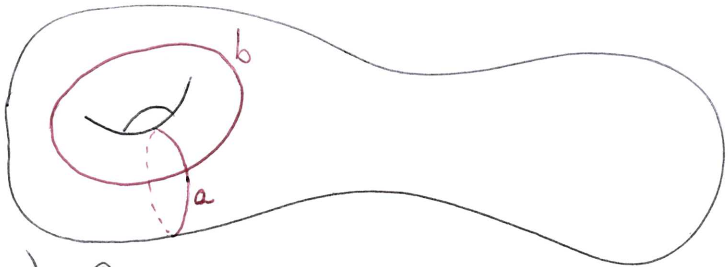
$\sim$ means isotopic

FACT: If b' & c' are closed geods in some hyper. metric on M then $i(b,c) = \#|b' \cap c'|$.

With this we can define $i_* : \mathcal{S} \longrightarrow [0,\infty)^{\mathcal{S}}$ & $I_* := \pi \circ i_*$.

$i_*(\mathcal{S}) \subset \mathbb{P}([0,\infty)^{\mathcal{S}})$ has a slightly surprising topology.

Claim: $i_*(D_a^n(b)) \longrightarrow i_*(a)$.

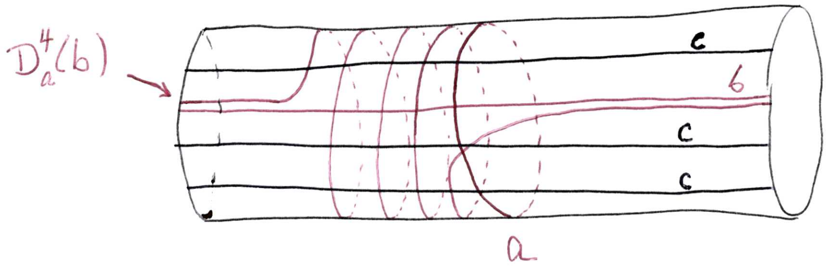


Pf: Consider $\frac{1}{n} i_*(D_a^n(b)) \subseteq [0,\infty)^{\mathcal{S}}$.

Pick $c \in \mathcal{S}$. Suppose $i(a,c)=0$.

Then $i(\frac{1}{n} D_a^n(b), c) = \frac{1}{n} i(b, D_a^{-n}(c)) = \frac{1}{n} i(b,c) \to 0$.

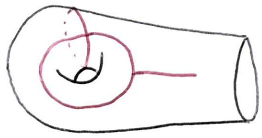
Next assume $i(a,c) = k > 0$. Examine a small annular nbd. of a.



in the picture $k=3$

$$\frac{1}{n} i(D_a^n(b), c) \leq \underbrace{\frac{1}{n} i(b,c)}_{\text{intersections outside the annulus about } a} + \underbrace{\frac{1}{n} \cdot i(na, c)}_{\text{intersections inside the annulus}} \longrightarrow k.$$

$$\therefore \quad \frac{1}{n} i_*(D_a^n(b)) \longrightarrow i_*(a) \quad \Rightarrow \quad i_*(D_a^n(b)) \longrightarrow i(a). \quad \boxtimes$$

We'll next describe an embedding $\mathcal{S} \to \mathcal{MF}$. Pick $c \in \mathcal{S}$. Choose a min'l graph $G \subset (M-c)$ such that $(M-c)$ is homeom. to a small neigh. of G in $M-c$. (Minimality ensures there are no spurious edges, e.g.  is not allowed.)

Then $M-G$ is an annulus. Choose a homeom. $(M-G) \longrightarrow [0,1] \times S^1$ taking c to $\{\frac{1}{2}\} \times S^1$. Use the homeo to pull back the "vertical" foliation of the annulus to $M-G$, so c is a closed leaf. Pull back the transverse measure of $[0,1]$. Add the singular leaves G to obtain a measured singular foliation $\mathcal{F}_c$. The choice of G was not canonical.

Prop: All choices of G result in Whitehead equivalent measured singular foliations.

(Pf in F-L-P 5.III.)

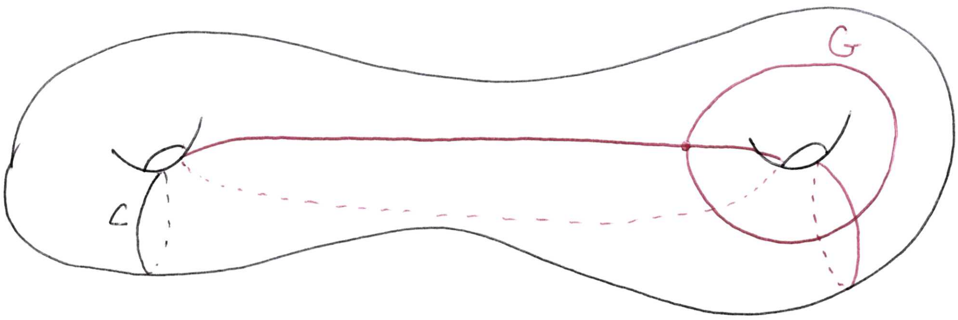
We therefore have a set map $\mathcal{S} \longrightarrow \mathcal{MF}$.

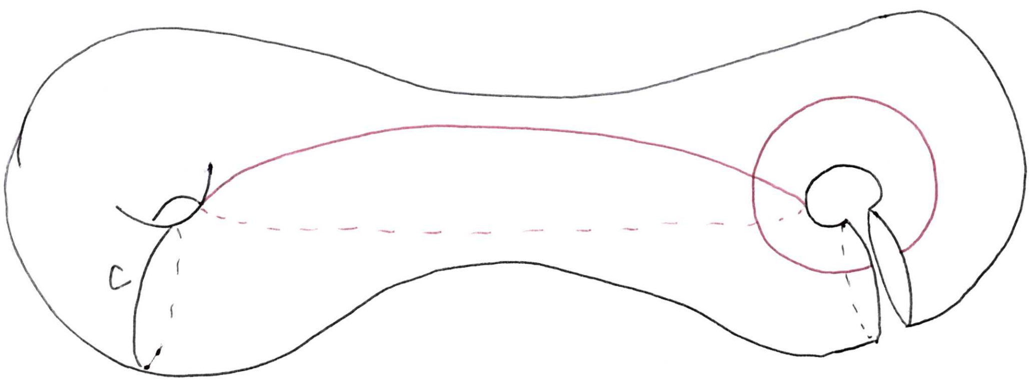
It's not too hard to believe this map is injective. In fact, we have the following commutative diagram of injections.

$$\begin{tikzcd}
\mathcal{S} \arrow[rr, "i_*"] \arrow[d] \arrow[dr, phantom, "\circ"] & & {[0,\infty)^{\mathcal{S}}} \\
\mathcal{MF} \arrow[urr, "I_*"'] & &
\end{tikzcd}$$

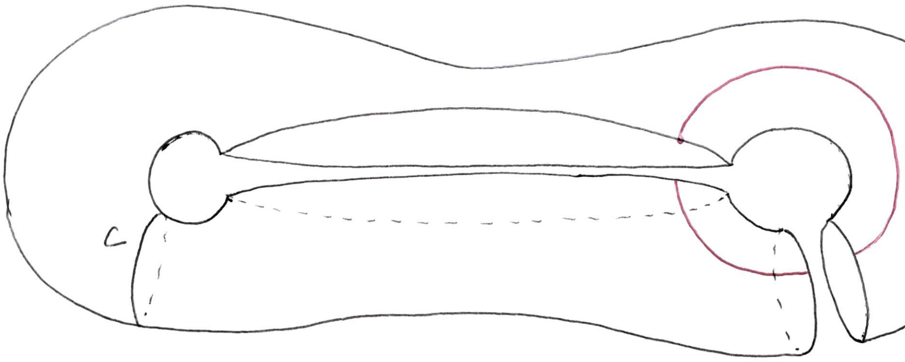
[FACT: $i_*(\mathcal{S}) \subset (\pi \circ I_*)(\mathcal{MF})$ is dense. Up to scaling, any meas. singular foliation is near a simple closed curve.]

Here are some pictures explaining how to get $\mathcal{F}_c$ from c.





cut



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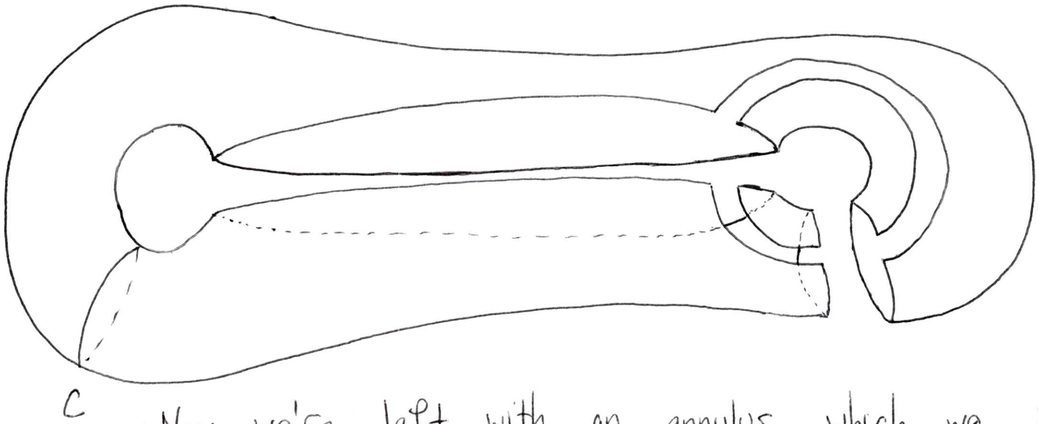
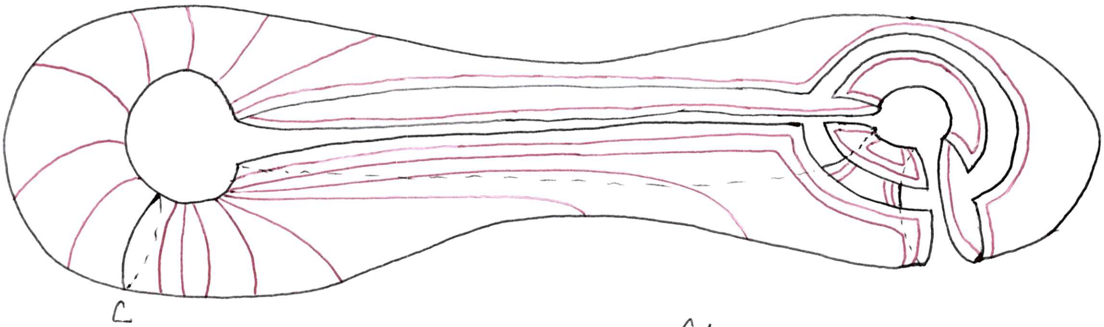
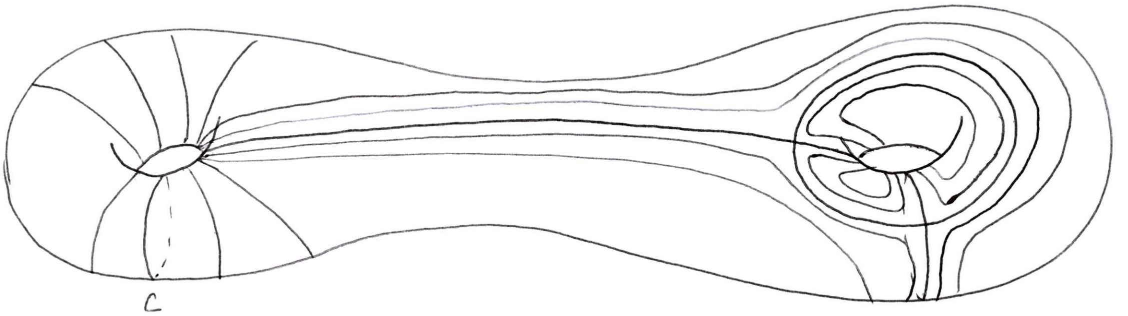


cut

Now we're left with an annulus, which we foliate.



Glue up to obtain $\mathcal{F}_c$



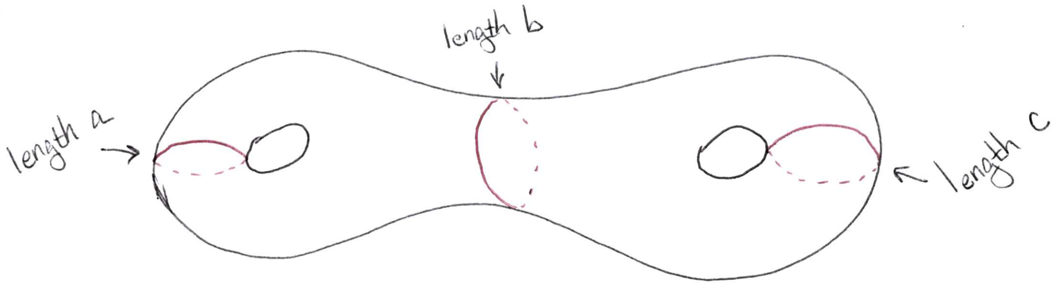
Claim: $\mathcal{S} \hookrightarrow \mathcal{PMF} \subset \mathbb{P}([0,\infty)^{\mathcal{S}})$ has dense image, i.e. for measured singular foliation $\mathcal{F}$ $\exists$ sequence $\{c_n\} \subset \mathcal{S}$ s.t. for any test curve $b \in \mathcal{S}$ we have

$$\alpha_n \cdot i(b, c_n) \longrightarrow I(\mathcal{F}, b)$$

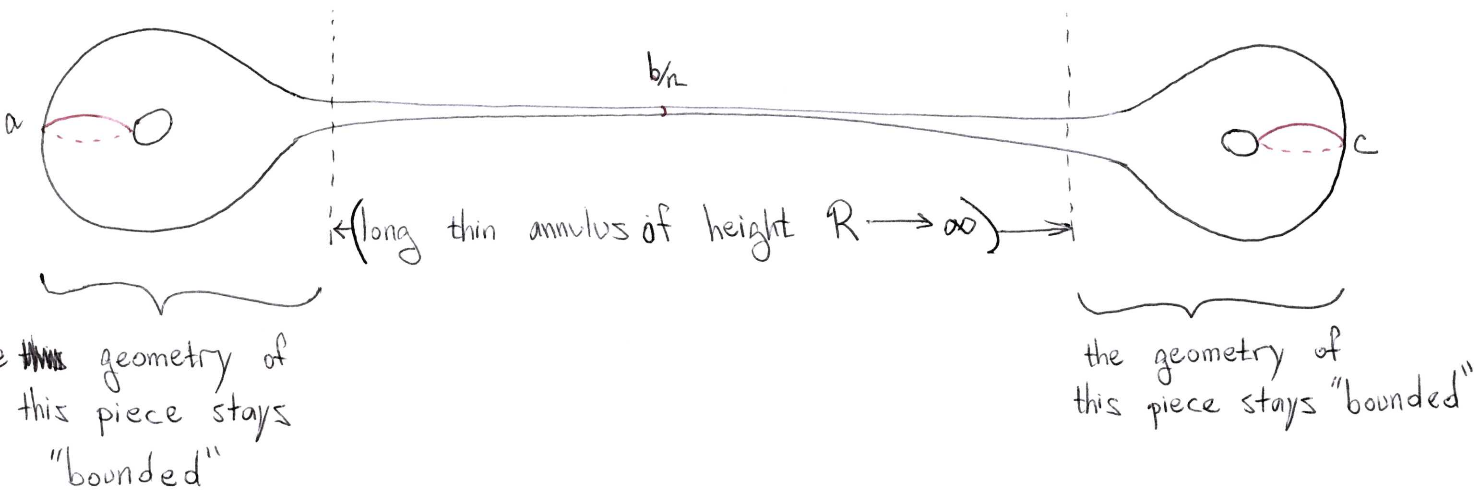
for some sequence of scaling factors α_n.

I won't prove this, at least not now.

How can a sequence of hyperbolic surfaces in $\mathcal{T}$ converge to a simple closed curve in $\mathbb{P}([0,\infty)^{\mathcal{S}})$? Let's give a simple example. Recall the Fenchel-Nielsen coordinates on $\mathcal{T}$. We have ~~$(X(\vec{0},\vec{0}), id) \in \mathcal{T}$ as a basepoint~~
$(X(a,b,c,\vec{0}), id) \in \mathcal{T}$ as a basepoint.



Consider the sequence $X_n := X(a, \frac{b}{n}, c, \vec{0})$. (Since we're not twisting, let's ignore markings, which will all be id.) A little hyperbolic geometry shows that for $n \gg 0$, X_n looks like



Let's call the curve of length b/n β. (Sorry for the bad notation.)

If $\alpha \in \mathcal{S}$ satisfies $i(\alpha,\beta) = 0$ then α will stay out of the annulus and $\ell_\alpha(X_n)$ remains bounded.

If $i(\alpha,\beta) = k > 0$ then $k \cdot R + \text{constant} \approx \ell_\alpha(X_n)$.

So consider the sequence $\frac{1}{R}\ell_*(X_n) \subset [0,\infty)^{\mathcal{S}}$.

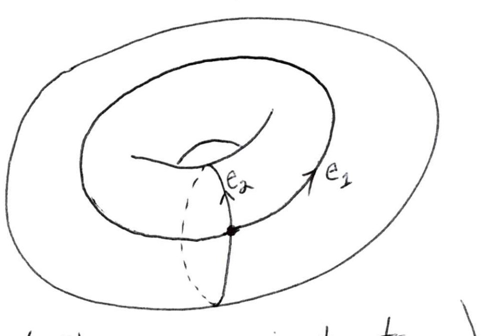
$$\frac{1}{R}\ell_\alpha(X_n) \longrightarrow i(\alpha,\beta) = i_*(\beta).$$

$$\Rightarrow \quad X_n \longrightarrow \beta \quad \text{in} \quad \mathbb{P}([0,\infty)^{\mathcal{S}}).$$

This is the simplest example of a seq. of hyper. surfaces converging to a scc.

The next goal is to show how this works for the humble torus T^2. Since flat tori can be scaled to produce new flat tori, we need a slightly different def'n of $\mathcal{Y}(T^2)$.

$$\mathcal{Y}(T^2) = \left\{ (X,m) \;\middle|\; \begin{array}{l} X \text{ flat 2-torus with orientable homeo} \\ m: T^2 \to X \ \text{ s.t. } \ \ell_{e_1}(X) = 1 \end{array} \right\}$$



$$= \left\{ (v_1,v_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \;\middle|\; \begin{array}{l} (v_1,v_2) \text{ positively oriented} \\ \text{and } \ v_1 = (1,0) \end{array} \right\}$$

clearly this is a dumb condition written this way

(It is convenient to orient e_i of T^2, but as elements of $\mathcal{S}$ they are unoriented.)

$$= \left\{ (x,y) \in \mathbb{R}^2 \mid y > 0 \right\}$$

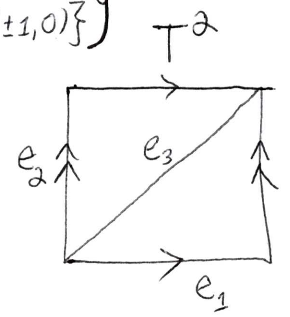
better

The bilipschitz metric I defined on $\mathcal{T}(T^2)$ works here, & reproduces the usual Euclidean topology.

What is $\mathcal{S}$?

$$\mathcal{S} = \left\{ \pm(m,n) \in (\mathbb{Z}\times\mathbb{Z})/_{\text{sign}} \;\middle|\; \begin{array}{l} \text{If } m \neq 0 \neq n \text{ then} \\ \gcd(m,n) = 1, \\ \text{Otherwise } (m,n) \in \{(0,\pm1),(\pm1,0)\} \end{array} \right\}$$

On T^2 define a 3rd scc e_3 as shown.



A curve $c \in \mathcal{S}$ can be written

$$c = \pm(m,n) = \pm(me_1 + ne_2)$$

(To make sense of this we think of $e_i \in \mathbb{R}^2$)

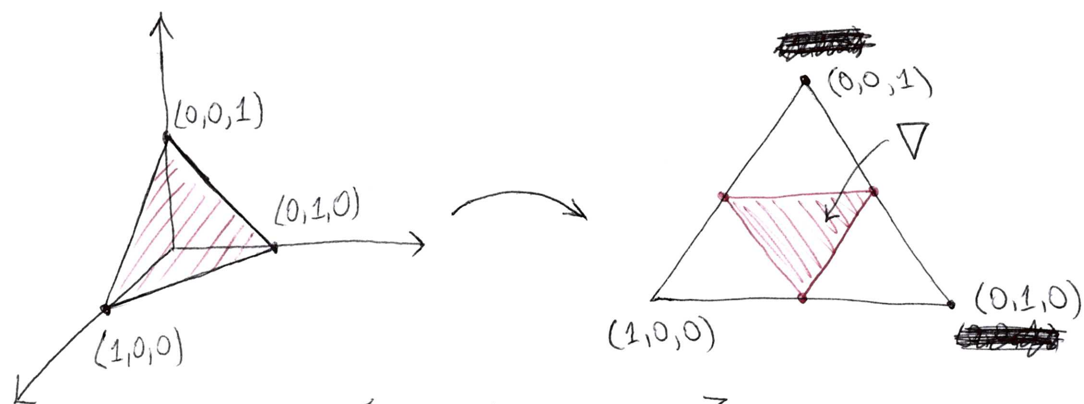
Exercise: $i(c,e_1) = |n|$, $i(c,e_2) = |m|$, $i(c,e_3) = |m-n|$.

Moreover, if $b = \pm(m'e_1 + n'e_2)$ then

$$i(b,c) = \left| \det\begin{pmatrix} m & n \\ m' & n' \end{pmatrix} \right|.$$

Notice any pair of numbers from $\{i(c,e_j)\}_{j=1}^{3}$ specifies two elements of $\mathcal{S}$, namely $\pm(m,n)$ and $\pm(m,-n)$. To obtain uniqueness all three numbers are required.

Consider the triangular regular simplex of $\mathbb{R}^3$:



In barycentric coords $\{(x,y,z) \mid x+y+z = 1\}$, the region where all 3 triangle inequalities $\left\{ \begin{array}{l} x+y \geq z \\ x+z \geq y \\ y+z \geq x \end{array} \right\}$ hold is shown on the right in red. Call it ∇.

Let $\mathrm{Cone}(\nabla) := \{ r \cdot (x,y,z) \mid (x,y,z) \in \nabla \text{ \& } r > 0 \}$

and $\mathrm{Cone}(\partial\nabla) := \{ r \cdot (x,y,z) \mid (x,y,z) \in \partial\nabla \text{ \& } r > 0 \}$.

Notice $\mathrm{Cone}(\partial\nabla)$ is the set of points where at least one triangle inequality is an equation.

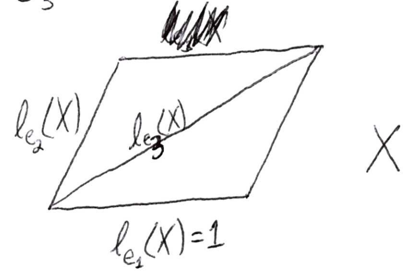
Define

$$\begin{aligned} i_3 : \mathcal{S} &\longrightarrow \mathrm{Cone}(\partial\nabla). \\ c &\longmapsto (i(e_1,c), i(e_2,c), i(e_3,c)). \end{aligned}$$

i_3 is injective. Also $\mathbb{Q} \cdot \mathrm{image}(i_3) = \mathbb{Q}^3 \cap \mathrm{Cone}(\partial\nabla)$.

So $\mathcal{S}$ forms the rational points of $\mathrm{Cone}(\partial\nabla)$.

Similarly
$$\begin{aligned} l_3 : \mathcal{Y} &\longrightarrow \mathrm{Cone}(\mathrm{int}(\nabla)) \\ X &\longmapsto (\underset{\substack{\uparrow \\ \text{This is always 1.}}}{l_{e_1}(X)}, l_{e_2}(X), l_{e_3}(X)) \end{aligned}$$



By trigonometry, l_3 is injective.

Let $\pi : [0,\infty)^3 \longrightarrow \mathbb{P}([0,\infty)^3) \approx \text{simplex}$ be projection.

(Here we're cheating slightly by silently ignoring the origin)

Then $\pi \circ l_3 : \mathcal{Y} \hookrightarrow \mathbb{P}([0,\infty)^3)$ is a homeom. onto the interior of ∇.

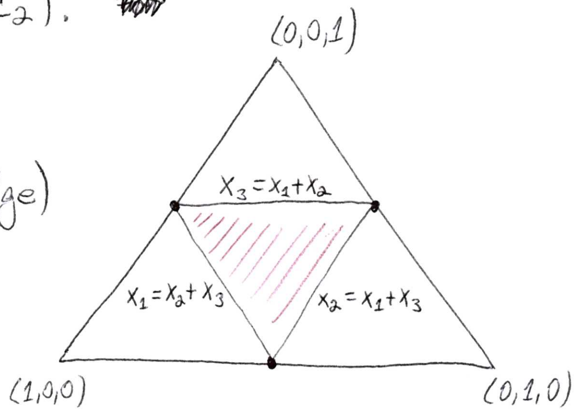
(This is again trigonometry.)

Lemma: For $c \in \mathcal{S}$ $\exists$ map $\varphi_c : \mathrm{cone}(\partial\nabla) \longrightarrow [0,\infty)$ s.t.

- φ_c is continuous
- $\varphi_c(\lambda \vec{x}) = \lambda\, \varphi_c(\vec{x}) \qquad (\lambda > 0)$
- $i(b,c) = \varphi_c\big(\, i(b,e_1),\ i(b,e_2),\ i(b,e_3)\big)$

Pf sketch: Let $c = \pm(m,n) = \pm(m e_1 + n e_2)$.

For $(x,y,z) \in \mathrm{cone}(\partial\nabla)$ define



$$\varphi_c(x,y,z) := \begin{cases} \left|\det\begin{pmatrix} y & -x \\ m & n \end{pmatrix}\right| & \text{if } x_3 = x_1 + x_2 \text{ (top edge)} \\[2ex] \left|\det\begin{pmatrix} y & x \\ m & n \end{pmatrix}\right| & \text{otherwise} \end{cases}$$

Let's check a case. If $m,n \geq 0$ and $b = \alpha e_1 + \beta e_2$ for $\beta \geq \alpha \geq 0$ then the barycentric coords of β are

$$\big(i(b,e_1),\ i(b,e_2),\ i(b,e_3)\big) = (\beta, \alpha, \beta - \alpha). \quad \text{So } x_3 \neq x_1 + x_2.$$

$$\varphi_c(\beta, \alpha, \beta-\alpha) = \left|\det\begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix}\right| = i(b,c). \quad \boxtimes$$

Use φ_c to define

$$\Phi : \mathrm{cone}(\partial\nabla) \longrightarrow [0,\infty)^{\mathcal{S}}$$
$$(x_1,x_2,x_3) \longmapsto \{ c \longmapsto \varphi_c(x_1,x_2,x_3) \}.$$

Φ is injective. Recall $i_* : \mathcal{S} \to [0,\infty)^{\mathcal{S}}$.

By construction $i_* = \Phi \circ i_3$.

$$\begin{tikzcd} & \mathrm{cone}(\partial\nabla) \arrow[d, "\Phi"] \\ \mathcal{S} \arrow[ur, "i_3"] \arrow[r, "i_*"'] \arrow[r, phantom, shift left=4ex, "\circ"] & {[0,\infty)^{\mathcal{S}}} \end{tikzcd}$$

Φ continuous $\Rightarrow$

$$(\pi \circ \Phi)(\mathrm{cone}(\partial\nabla)) \subset \overline{\mathcal{S}} \subset \mathbb{P}([0,\infty)^{\mathcal{S}}).$$

Claim: The closure of $\mathcal{S}$ in $\mathbb{P}([0,\infty)^{\mathcal{S}})$, i.e. $\overline{(\pi \circ i_*)(\mathcal{S})}$, equals the image $(\pi \circ \Phi)(\text{cone}(\partial\nabla))$.

Pf: Suppose $\{c_n\} \subset \mathcal{S}$ s.t. $(\pi \circ i_*)(c_n)$ converges. Then $\exists$ $\lambda_n > 0$ s.t. $\lambda_n i_*(c_n)$ converges to (nonzero) $g \in [0,\infty)^{\mathcal{S}}$.

$\Rightarrow$ $\forall$ $b \in \mathcal{S}$, $\lambda_n i(c_n, b) \longrightarrow g(b)$.

We want to show $g \in \text{image}(\Phi)$.

$$g(b) = \lim \lambda_n i(c_n, b) = \lim \lambda_n \varphi_b(i(c_n, e_1), i(c_n, e_2), i(c_n, e_3))$$

$$= \varphi_b(g(e_1), g(e_2), g(e_3)) = \Phi(g(e_1), g(e_2), g(e_3)).$$

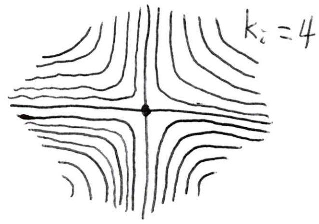
Must check that $(g(e_1), g(e_2), g(e_3)) \in \text{Cone}(\partial\nabla)$, but OK. $\boxtimes$

This shows $\overline{\mathcal{S}} \approx S^1$ in $\mathbb{P}([0,\infty)^{\mathcal{S}})$.

What about singular measured foliations on T^2?

Let's take a minute to back up and study them.

Thm (Euler-Poincaré Formula): Let M be a closed surface with a measured singular foliation $\mathcal{F}$ and singular set $S = \{p_1, p_2, \dots, p_n\}$. For each p_i let $k_i \in \{3, 4, 5, \dots\}$ be the number of leaves coming out of p_i. (For example, in the picture $k_i = 4$.) Then



$$2\chi(M) = \sum_{i=1}^{n} (2 - k_i).$$

Pf: (See F-L-P, Ch 5.I.6.).

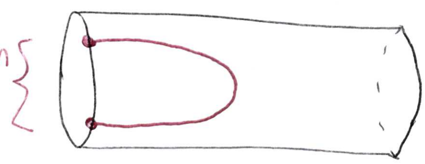
Cor: If M is a torus then $S = \emptyset$.

Let's examine the torus case. Let $M = T^2$. Fix $\mathcal{F}$ on T^2.

FACT: $\exists$ homotopically nontrivial s.c.c. $c \subset T^2$ transverse to $\mathcal{F}$.
"Pf": Look for a recurrent orbit in the transverse direction. $\boxtimes$

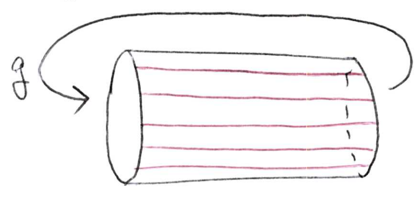
Cut T^2 along c to get an annulus: $\mathcal{F}$ is everywhere transverse to c $\Rightarrow$ a leaf cannot begin and end on the same side of the annulus.

Some pt. in this interval would not be transverse

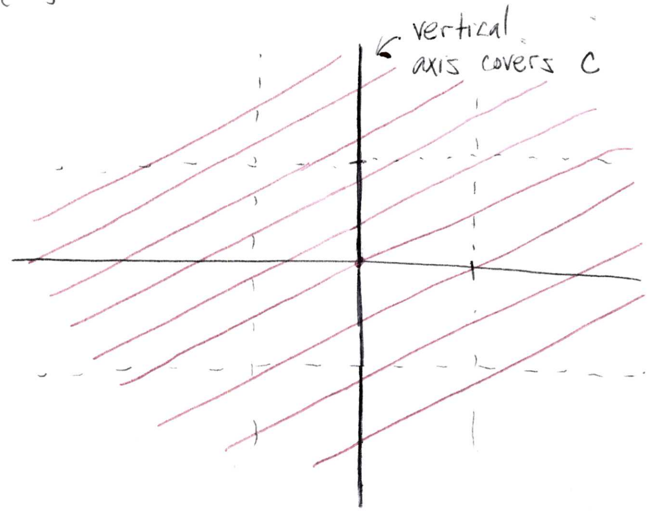


not possible

So (with some work) one can show the annulus is homeom. to $[0,1] \times S^1$ with leaves $[0,1] \times \{\theta\}$. A gluing homeom. $g: S^1 \to S^1$ recovers T + $\mathcal{F}$. After possibly reparametrizing

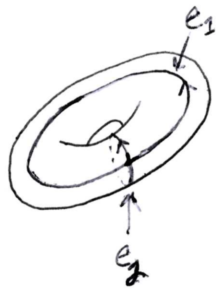


the S^1 factor we can assume g is rotation by some angle θ_0. After gluing by g we obtain coords. $S^1 \times S^1$ for T^2 where $c = \{0\} \times S^1$. In these coords, the universal cover looks like:

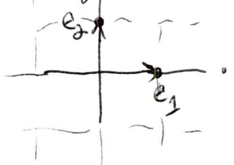


The leaves are straight lines with slope $\theta/2\pi$.

Suppose T^2 can equipped with a marking



and universal cover



. $\exists$ linear map conjugating one universal cover to the other taking c to e_2. So $\mathcal{F}$ will be taken to straight lines in the $\{e_1, e_2\}$ coordinate system also.

Conclusion: $\mathcal{F}$ is isotopic to a foliation on $T^2 = \mathbb{R}^2 / \langle (1,0), (0,1) \rangle$ with straight leaves.

So as a set, and ignoring measure, ~~the~~ foliations on T^2 up to isotopy is just $\mathbb{RP}^1 = \left\{ \begin{matrix} \text{slopes} \\ \text{on } T^2 \end{matrix} \right\}$.

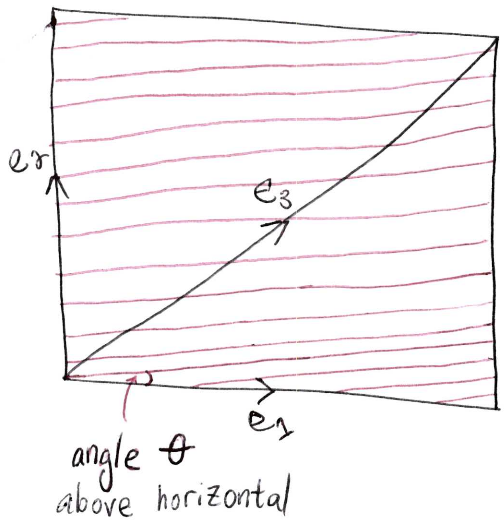
The measure of $\mathcal{F}$ is just determined by the induced measure on the transverse curve c. After possibly another coordinate change we may assume this measure is $|d\theta|$, so it's only invariant is the total mass. Thus, as a set

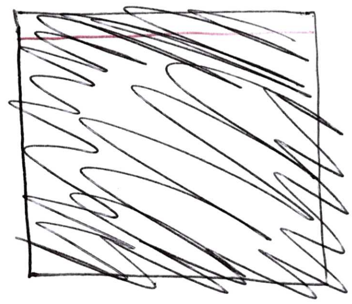
$$\mathcal{MF} = (0, \infty) \times \mathbb{RP}^1.$$

How did this picture of $\mathcal{MY}$ fit into our previous framework (e.g. i_3, l_3, Φ)? Define

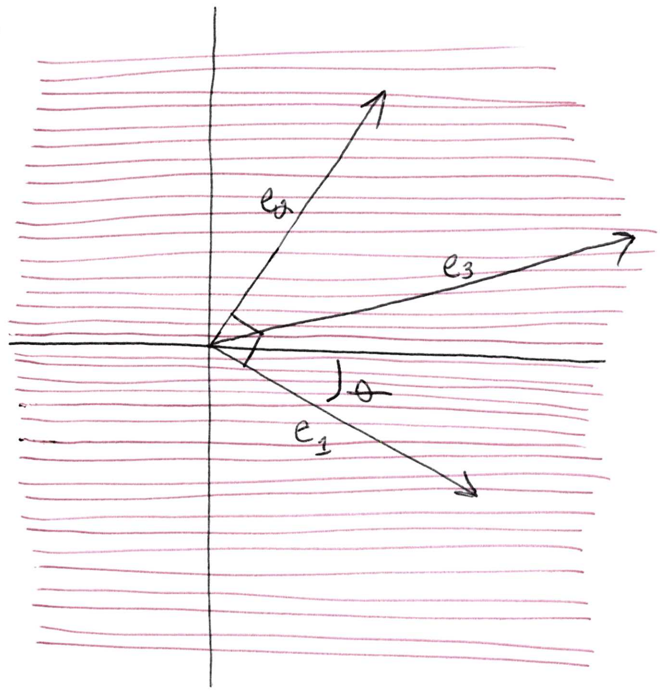
$$I_3 : \mathcal{MY}(T^2) \longrightarrow \mathrm{Cone}(\partial\nabla)$$
$$\mathcal{Y} \longmapsto (I(\mathcal{Y}, e_1), I(\mathcal{Y}, e_2), I(\mathcal{Y}, e_3))$$

where $\mathcal{Y}$ is as shown:





It's easier to compute $I(\mathcal{Y}, e_i)$ if we tilt this picture, making $\mathcal{Y}$ horizontal:



Then

$I(\mathcal{Y}, e_1) = |\sin\theta|$

$I(\mathcal{Y}, e_2) = |\cos\theta|$

$I(\mathcal{Y}, e_3) = |\cos\theta - \sin\theta|$.

So I_3 has image in $\mathrm{Cone}(\partial\nabla)$, as claimed.

Recall the topology on $\mathcal{MY}$ was defined by assuming the $I(\mathcal{Y}, \cdot)$ fcts are continuous. So I_3 is, in fact, a homeomorphism.

Recall $\Phi : \mathrm{cone}(\partial\nabla) \longrightarrow [0,\infty)^{\mathcal{S}}$, &

our thickening construction $\mathcal{S} \xrightarrow{t} \mathcal{MY}$.

How do $I_3 \circ t$ & i_3 relate? Fix $c = \pm(m,n) \in \mathcal{S}$

& $t(c) = \mathcal{Y}_c$. Then the angle of $\mathcal{Y}_c$ is $\theta = \arctan\left(\frac{n}{m}\right)$.

So
$$\begin{aligned} I_3(\mathcal{Y}_c) &= (|\sin\theta|, |\cos\theta|, |\cos\theta - \sin\theta|) \\ &= \frac{1}{\sqrt{n^2+m^2}}(|n|, |m|, |m-n|), \end{aligned}$$

& we see $(I_3 \circ t)(c) = \sqrt{n^2+m^2} \cdot i_3(c)$. They're projectively equivalent. This gives the following commutative diagram:

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "i_3"] \arrow[dd, "t"] \arrow[rr, bend left, "i_*"] & \mathrm{Cone}(\partial\nabla) \arrow[r, "\Phi"] & {[0,\infty)^{\mathcal{S}}} \arrow[dr, "\pi"] & \\
& & & \mathbb{P}([0,\infty)^{\mathcal{S}}) \\
\mathcal{MY} \arrow[r, "I_3"] \arrow[rr, bend right, "I_*"'] & \mathrm{Cone}(\partial\nabla) \arrow[r, "\Phi"] & {[0,\infty)^{\mathcal{S}}} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Lecture 4 2009-04-21

Introduction to the curve complex of the torus.

Let $\mathcal{S}$ be the set of isotopy classes of unoriented homot. nontrivial simple closed curves on the torus T^2. Recall

$$\mathcal{S} = \{\pm(0,1), \pm(1,0)\} \cup \{\pm(m,n) \mid \gcd(m,n)=1\} \subset \frac{\mathbb{Z}\times\mathbb{Z}}{\pm 1},$$

and $i(\pm(m,n), \pm(m',n')) = \left|\det\begin{pmatrix} m & m' \\ n & n'\end{pmatrix}\right|$.

boundary of the upper half-plane

Imagine $\mathcal{S} = \mathbb{P}\mathbb{Q} = \{\frac{p}{q} \in \mathbb{Q}\cup\{\infty\} \mid \gcd(p,q)=1\} \subset \partial \mathbb{H}^2 = \mathbb{R}\cup\{\infty\}$.

Turn $\mathcal{S}$ into a graph. Add an edge between $\pm(m,n)$ & $\pm(m',n')$ iff $i(\pm(m,n), \pm(m',n')) = 1$, i.e. iff $\exists$ a homeo of T^2 taking our pair to $e_1=(1,0)$ & $e_2=(0,1)$.

Note that any two distinct classes in $\mathcal{S}$ have positive intersection, so intersecting exactly once is minimal.

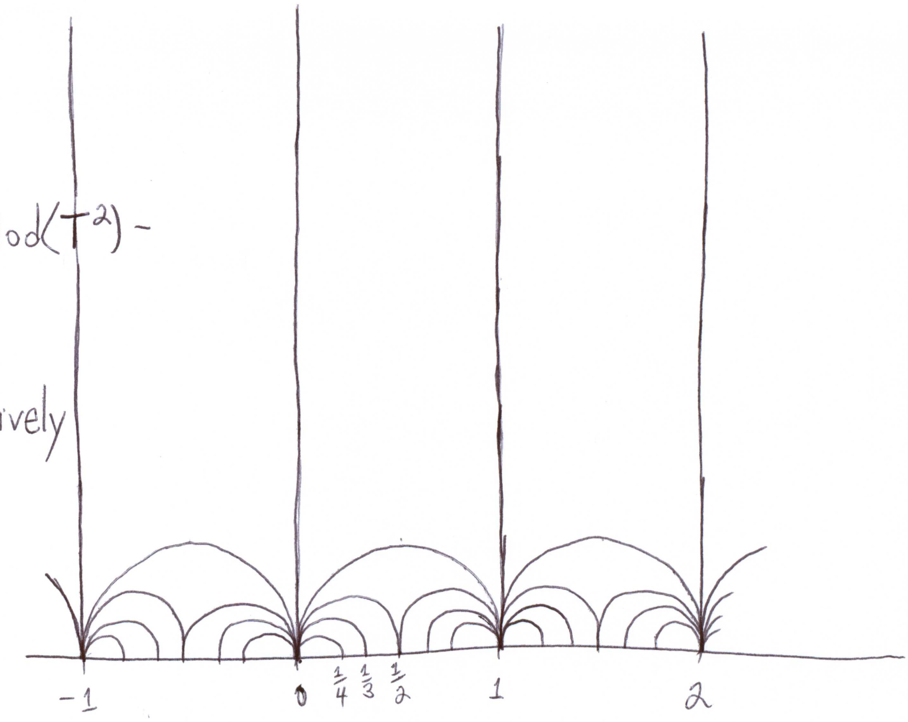
Call the resulting graph $\mathcal{C}(T^2)$, the graph of curves on T^2. $\mathcal{C}(T^2)$ is also known as the Farey graph.

Embed $\mathcal{C} = \mathcal{C}(T^2)$ into the upper half-plane $\mathbb{H}^2$ by making the edges bi-infinite geodesics between points in $\mathcal{S} = \mathbb{P}\mathbb{Q} \subset \mathbb{R}$.

The curve $\pm(1,0) = \infty$ is joined to $\begin{pmatrix} 1 & m \\ 0 & n\end{pmatrix} = \pm 1$

$\Rightarrow \pm(m,n) = \pm(m,1) = m \in \mathbb{Z}$. Similarly, $\pm(0,1) = 0$ is joined to $\pm(m,n) = \pm(1,m) = \frac{1}{m}$.

A finite piece of $\mathcal{C} \subset \overline{\mathbb{H}^2}$ looks like:



By def'n, $\mathcal{C}$ is $PSl_2\mathbb{Z} = Mod(T^2)$-invariant.

$PSl_2\mathbb{Z}$ acts transitively on $\mathbb{PQ}$, implying $\mathcal{C}$ is homogeneous. (Well, at least all the vertices look the same.)

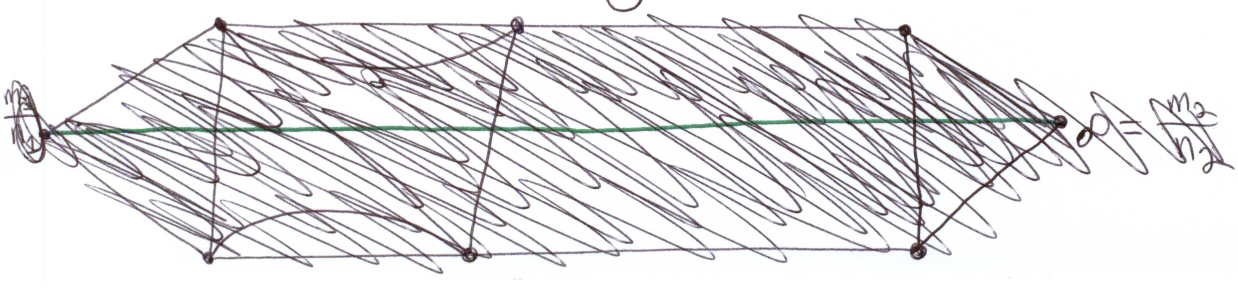
The stabilizer of $\infty = \pm(1,0)$ equals $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, which are the Dehn twists along $\infty = \pm(1,0) = e_1$.

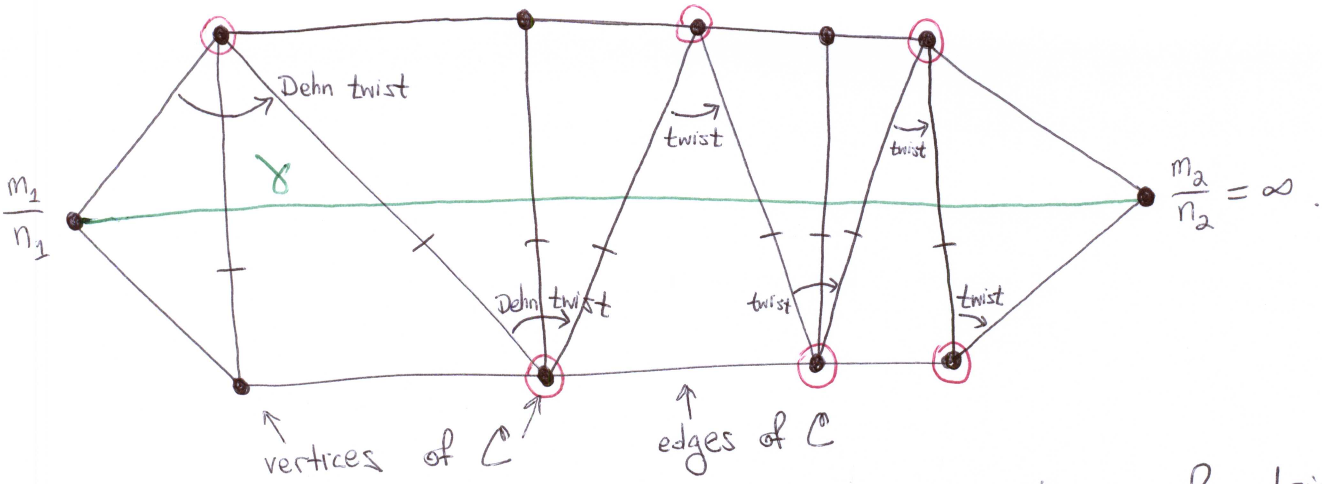
Similarly, the stabilizer of $\frac{p}{q} = \pm(p,q)$ are the Dehn twists about $\pm(p,q)$.

Note $\mathbb{C}$ is locally infinite.

Claim: $\mathbb{C}$ is connected.

Pf: Let γ be the hyperbolic geod. from $\pm(m_1,n_1) = \frac{m_1}{n_1} \in \mathbb{PQ}$ to $\pm(m_2,n_2) = \frac{m_2}{n_2} \in \mathbb{PQ}$. Using an elt. of $PSl_2\mathbb{Z}$ we may assume $\frac{m_2}{n_2} = \infty$. Then γ is a vertical line. $\mathbb{C}$ cuts $\mathbb{H}^2$ into ideal triangles. Combinatorially, γ hits these triangles as shown:





Once you see that only a finite number of triangles can hit γ, then connectivity follows by simply pushing γ onto the 1-skeleton of the triangles. ☒

Claim: $\mathcal{C}$ has infinite diameter.

Pf: Each of the edges of $\mathcal{C}$ crossed by γ separate $\mathcal{C}$. So any path from $\frac{m_1}{n_1}$ to $\frac{m_2}{n_2}$ must traverse the edges marked with ticks, possibly at a vertex. ~~so the distance in $\mathcal{C}$ between $\frac{m_1}{n_1}$ & $\frac{m_2}{n_2}$ is at least [illegible]. Push this idea farther to obtain any points arbitrarily far apart, using that Dehn twists are stabilizers~~ ☒ Going from $\frac{m_1}{n_1}$ to $\frac{m_2}{n_2}$ in $\mathrm{Mod}(T^2)$ involves doing Dehn twists about the curves represented by the vertices circled in red, the so-called "pivots". Each pivot adds 1 to the distance. So path with many pivots must be very long. ☒

Notice performing a large number of Dehn twists at a fixed pivot does not increase the distance.

(Reference: "A geometric approach to the complex of curves on a surface" by Y. Minsky.)

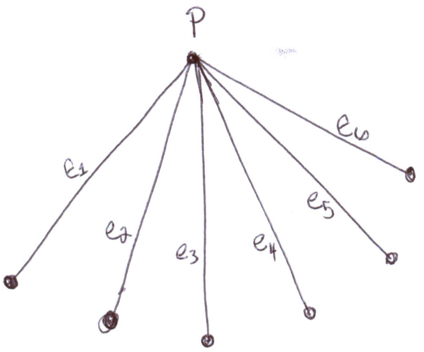
Let's formalize these ideas a little.

Let v_+ and v_- be distinct points in $\mathbb{R} \cup \{\infty\}$. If $v_* \in \mathbb{Q} \cup \{\infty\}$ then think of it as an elt. of $\mathcal{S}$.

~~Join v_+ & v_- by a hyperbolic geodesic~~

Let $E(v_+, v_-)$ denote the set of edges of $\mathcal{C}$ separating v_+ & v_-. Define an order on $E(v_+, v_-)$:

$e < f \iff e$ separates v_- from the interior of f. A <u>pivot</u> is a vertex of $\mathcal{C}$ shared by ≥ 1 pair of consecutive edges of $E(v_+, v_-)$. (The pivots are circled in red in the picture on the previous page.) A <u>block</u> of pivot p is a max'l set of consecutive edges $e_1 < e_2 < \cdots < e_{w(p)}$ all sharing the vertex p. $w(p)$ is the width of the block.

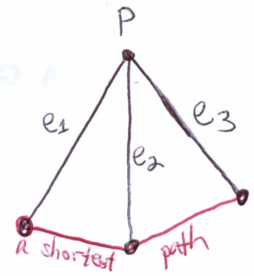


a block of width 6

~~shortest path from v_- to v_+~~

Assume for now that $v_+, v_- \in \mathcal{S}$. A shortest path from v_- to v_+ must intersect each edge of a block, possibly through a vertex. If $w(p) > 3$ then a shortest path will intersect the block in e_1 and/or

$e_{w(p)}$ only. In all cases a shortest path will first intersect the block, stay within a (closed) $\frac{1}{2}$-neighborhood of the block, and leave the block.

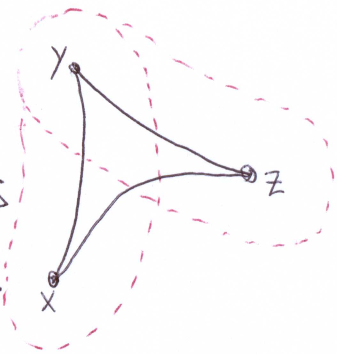


If $w(p) \leq 3$ then a shortest path may avoid p.

Prop. (Minsky?): $\mathcal{C}$ is Gromov hyperbolic.

Recall the def'n.

Def: Let X be a geodesic metric space. X is Gromov hyperbolic if $\exists$ $\delta > 0$ with the following property: Join $x, y, z \in X$ by shortest paths. Then the path from x to z is contained in the union of the δ-neighborhoods of the paths from x to y & y to z.



Pf of Proposition: It suffices to consider a triple $x, y, z \in \mathcal{S}$ & shortest paths $[xy]$, $[yz]$, & $[xz]$ in $\mathcal{C}$.

Let e be an edge of $E(x,z)$.

Either: (i) $e \in E(x,y)$ or

(ii) $e \in E(y,z)$ or

(iii) y is a vertex of e.

In case (i), $[xy]$ must hit e.

In case (ii), $[yz]$ must hit e.

In case (iii), y is in e.

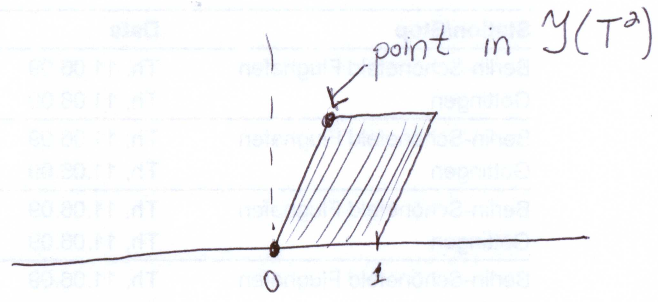
So e is inside a (closed) 1-neigh. of $[xy] \cup [yz]$. This implies the vertices of $[xz]$ are in a closed 1-neigh of $[xy] \cup [yz]$. ~~Thinking about the first or last edges of [xz]~~ we see that $[xz]$ is inside a (closed) $\frac{3}{2}$-neigh. of $[xy] \cup [yz]$. $\boxtimes$

Being Gromov hyperbolic is good for many reasons. Consider a pair of infinite hyperbolic geodesics beginning at 0 and terminating at $r_1, r_2 \in \mathbb{R} - \mathbb{Q}$. These geods. determine ordered edge sequences $\{e_n^1\}$ + $\{e_n^2\}$. As soon as the sequences $e_1^1, e_2^1, e_3^1, \ldots$ + $e_1^2, e_2^2, e_3^2, \ldots$ see an unequal pair, then the corresponding geodesics in $\mathcal{C}$ will begin to diverge linearly. (From the model on page 4 for shortest paths we see shortest paths are unique except for finite ambiguity in traversing blocks of width ≤ 3.) From this it follows that the Gromov boundary of $\mathcal{C}$ is $\mathbb{R} - \mathbb{Q}$. Notice that $\mathcal{C} \cup \partial \mathcal{C}$ looks like $\mathbb{R} \cup \{\infty\}$, but the topology is very different. In particular, it is not compact. For example, the sequence $\{n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ has no convergent subseq.. In a sense, this is the only way a sequence can diverge: by having lots of twisting. An element of $\partial \mathcal{C} = \mathbb{R} - \mathbb{Q}$ corresponds to a measured foliation of irrational slope.

Discuss axes of hyperbolic elements of $PSL_2\mathbb{Z} = \mathrm{Mod}\,T^2$.

What's the connection between $\mathcal{C}(T^2)$ and $\mathcal{Y}(T^2)$.

Recall $\mathcal{Y}(T^2) = \{(x,y) \mid y > 0\}$.



For $c \in \mathcal{S}$ let

$$U_c := \left\{ (x,y) \in \mathcal{Y}(T^2) \;\middle|\; \frac{\text{length}(c)}{\text{area}} \leq \varepsilon \right\}.$$

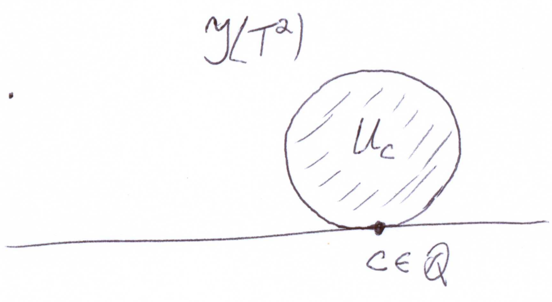
$= \begin{pmatrix} \text{flat tori where } c \\ \text{is not long} \end{pmatrix}$

ε won't be very small, just a medium size

For example
$$U_\infty = \left\{ (x,y) \;\middle|\; \frac{\text{length}(\infty)}{\text{area}} = \frac{\text{length}(\pm(1,0))}{\text{area}} = \frac{1}{y} \leq \varepsilon \right\}$$

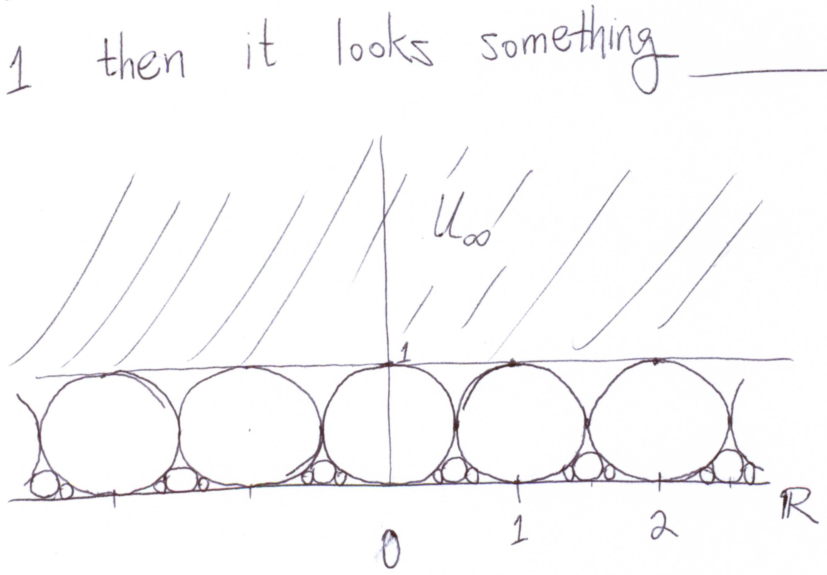
$$= \left\{ (x,y) \;\middle|\; y \geq \frac{1}{\varepsilon} \right\}.$$

Apply $\mathrm{Mod}(T^2) = PSL_2\mathbb{Z}$ to U_∞ and conclude that U_c for $c \neq \infty$ is a horodisk tangent to c.



FACT: $\sup\limits_{\varepsilon} \{ U_c \text{ are pairwise disjoint} \} = 1$.

If $\varepsilon = 1$ then it looks something like



Consider the nerve of the collection of closed sets $\{U_c\}_{c\in\mathcal{S}}$.
This nerve is a graph with vertices U_b & U_c joined by an edge iff $U_b \cap U_c \neq \emptyset$. (The higher skeleta are empty.)
This nerve is exactly $\mathcal{C}(T^2)$. This is not an accident. The collection $\{U_c\}_{c\in\mathcal{S}}$ is often called the thin parts of ~~moduli~~ Teichmüller space. Then $\mathcal{C}$ is the nerve of the thin parts. This will persist in higher genera.

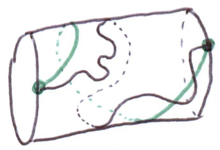
Lecture 5

Subsurface projections in the case of the torus

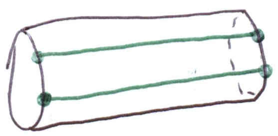
Let $\mathcal{S}$, as usual, be the set of isotopy classes of (embedded) homotopically nontrivial simple closed curves. Recall $\mathcal{S}$ is naturally identified with $\mathbb{P}\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$. For $c \in \mathcal{S}$, $T^2 - c$ is an annulus. We need the def'n of the curve complex of an annulus, which is annoyingly complex. (No pun intended.) For the sake of culture I'll give the official def'n from Masur-Minsky's "Geometry of the complex of curves. II." For annulus A, ~~let the vertices~~ a compact annulus with boundary, let the vertices of $\mathcal{C}(A)$ be the set

$$\left\{ \begin{array}{l} \text{paths from one boundary} \\ \text{component of } A \text{ to the other} \end{array} \right\} \Big/ \left\{ \begin{array}{l} \text{homotopies } \underline{\text{fixing}} \\ \text{the boundary pointwise} \end{array} \right\}.$$

E.g. These are the same vertex:



These are not:



Obviously this set $\mathcal{C}(A)$ is huge, but we're stuck with it.

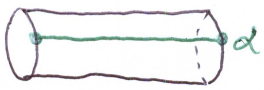
Join the vertices of $\mathcal{C}(A)$ by an edge if they have representatives with disjoint interiors, forming the curve complex of the annulus.

For $\alpha, \beta \in \mathcal{C}(A)$ vertices, the signed algebraic intersection number $\alpha \cdot \beta$ is well defined. (Only count interior intersections.)

Lemma: $d_{\mathcal{C}(A)}(\alpha, \beta) = 1 + |\alpha \cdot \beta|$

Lemma: $\gamma \cdot \alpha = \gamma \cdot \beta + \beta \cdot \alpha + \varepsilon$ for $\varepsilon \in \{-1, 0, 1\}$.
(Note $\gamma \cdot \alpha = -(\alpha \cdot \gamma)$!)

Picking a base $\alpha \in \mathcal{C}(A)$ defines a map $f : \mathcal{C}(A) \to \mathbb{Z}$



$\beta \longmapsto \beta \cdot \alpha.$

Lemma: f is a quasi-isometry, namely

$$|f(\beta) - f(\gamma)| \leq d_{\mathcal{C}(A)}(\beta, \gamma) \leq |f(\beta) - f(\gamma)| + 2.$$

So without losing anything one can imagine $\mathcal{C}(A)$ is simply $\mathbb{Z}$ (as a metric space, not a group).

(curves on T^2)

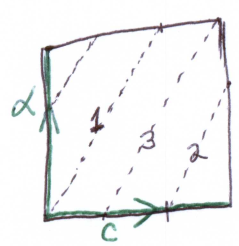
Given $c \in \mathcal{S}$ let A_c be the compact annulus obtained in the obvious way from $T - c$. For $b \in \mathcal{S} - \{c\}$ define the projection $\pi_c(b)$ to $\mathcal{C}(A_c)$ as the set:

$$\left\{ \beta \;\middle|\; \begin{array}{l} \beta \text{ equals the restriction to } A_c \text{ of} \\ b' \text{ for some s.c.c. } b' \text{ isotopic to } b \end{array} \right\} \subset \mathcal{C}(A_c).$$

Then $\pi_c(b)$ is a set of diameter 1.

To make this more explicit, let $c = \pm(1,0) = \frac{1}{0} = \infty \in \mathcal{S}$ and $\alpha = \pm(0,1) = 0 \in \mathcal{S}$.

Then $b = \pm(p,q) = \frac{p}{q} \in \mathcal{S}$ will project to the annulus A_c as q curves $\{b_i\}$ of slope q/p. (I guess we should have taken reciprocals somewhere.) Each b_i will intersect α either $\lfloor p/q \rfloor$ or $\lceil p/q \rceil$ times. So the projection π_c can safely be thought of as taking the integer part of p/q.

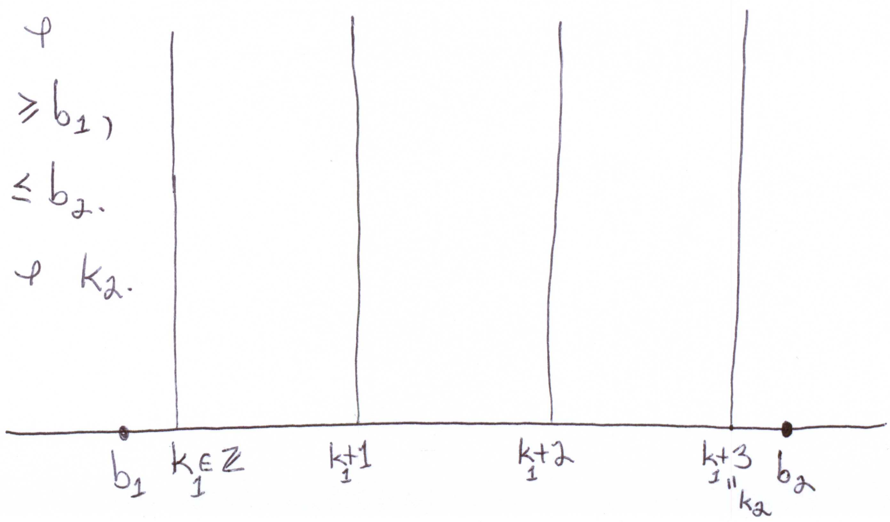


The dashed line is the $\pm(2,3) = \frac{2}{3}$ curve. It has 3 components in A_c.

Prop (Bounded Geodesic Image): Suppose g is a geodesic in $\mathcal{C}(T^2)$ disjoint from $c \in \mathcal{S}$. Then the diameter of the projection $\pi_c(g)$ is at most 4.

Pf: Wolog assume $c = \frac{1}{0} = \infty$. Assume $\exists\ b_1, b_2 \in g$ s.t. the diameter of $\pi_c(b_1 \cup b_2)$ is ≥ 5. Then $b_1 = \frac{p_1}{q_1}$ & $b_2 = \frac{p_2}{q_2}$ are separated by at least 4 integer points.

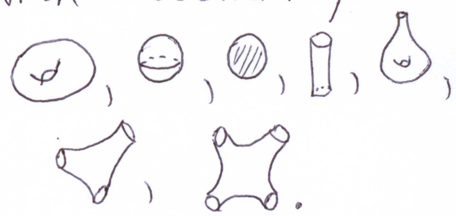
Suppose wolog $b_1 < b_2$ & k_1 is the least integer $\geq b_1$, & k_2 " " greatest " $\leq b_2$. g must pass through k_1 & k_2.



The unique geodesic from k_1 to k_2 is $\{k_1, \infty, k_2\}$. By assumption g is disjoint from ∞, yielding a contradiction. $\boxtimes$

This proposition is true in higher genus, proved by Masur-Minsky.

The gen'l curve complex

We will need to allow surfaces with boundary. Let M be a compact oriented surface, not .

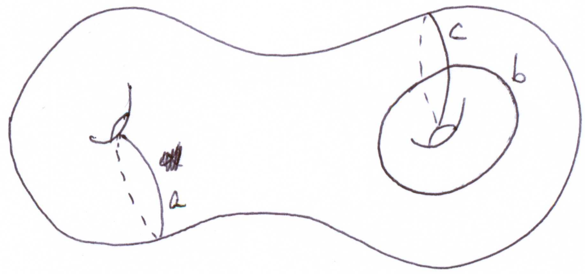
$\mathcal{S} = \left\{ \begin{array}{l} \text{isotopy classes of simple closed} \\ \text{curves not isotopic into } \partial M \end{array} \right\}$.

Then $C(M)$ is a simplicial complex with a $(k-1)$-simplex given by a pairwise disjoint k-tuple of $\mathcal{S}$. Make it a metric space by making each simplex a standard Euclidean simplex & using a path metric. We will only consider the 1-skeleton, the curve graph.

Claim: The curve graph is locally infinite.

Pf: Find $a, b, c \in \mathcal{S}$ s.t. $a \cap b = a \cap c = \emptyset$ & $b \cap c \neq \emptyset$.

(This is possible because we ruled out the "low genus" cases, & spheres with < 5 punctures.)



Then the curves $\{D_b^n(c)\}_{n\in\mathbb{Z}}$ are all distance 1 from $a \in \mathcal{S}$. ⊠

Claim: The curve graph is connected. In fact

$$d(\alpha,\beta) \leq 2\cdot i(\alpha,\beta) + 1.$$

Pf: Assume $\#(\alpha\cap\beta) = i(\alpha,\beta)$, i.e. they intersect minimally.

If $i(\alpha,\beta)=0$ then we're done.

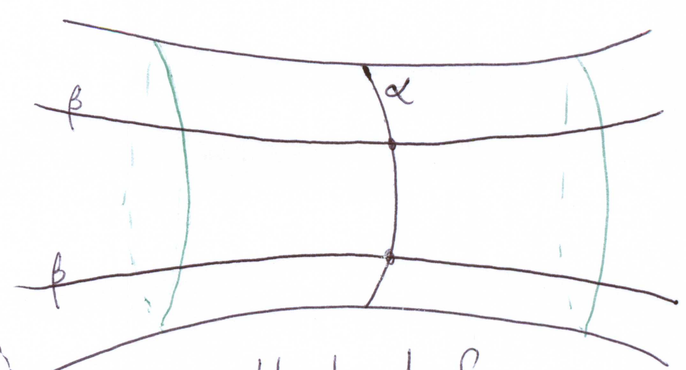
If $i(\alpha,\beta)=1$ then consider a small neighborhood $\mathcal{U}$ of $\alpha\cup\beta$. $\mathcal{U}$ is necessarily a punctured torus. Consider the curve $\partial\mathcal{U}$. If $\partial\mathcal{U}$ is isotopic into ∂M the M is a punctured torus. We assumed M is <u>not</u> a punctured torus. $\Rightarrow$ $\partial\mathcal{U} \in \mathcal{C}(M)$, $d(\partial\mathcal{U},\alpha) = d(\partial\mathcal{U},\beta) = 1$

$\Rightarrow$ $d(\alpha,\beta) = 2$.

Now assume $i(\alpha,\beta) = k \geq 2$ and argue by induction.

Consider a pair of adjacent points in $\alpha\cap\beta$.

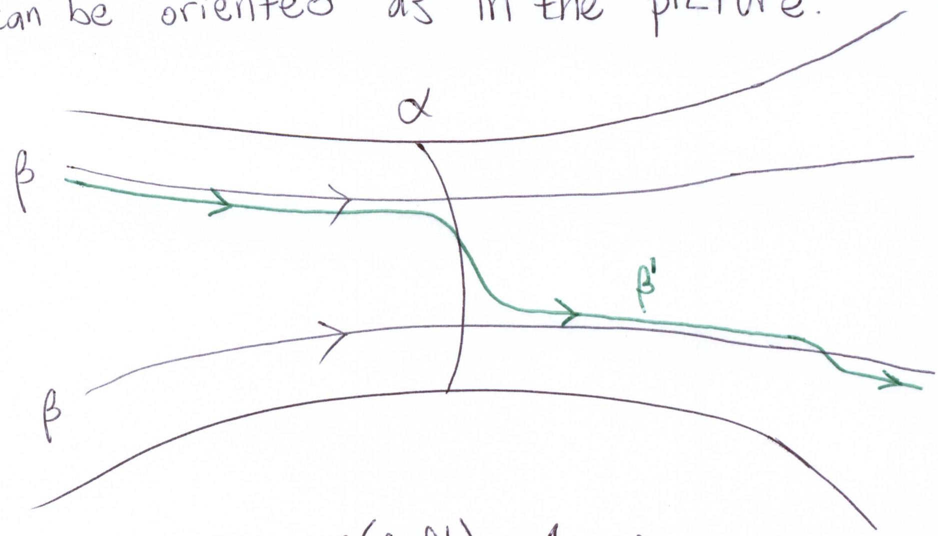
[redacted]



a neighborhood of α

[redacted]

Case I: Assume β can be oriented as in the picture:



Then do a surgery as shown to produce β'.

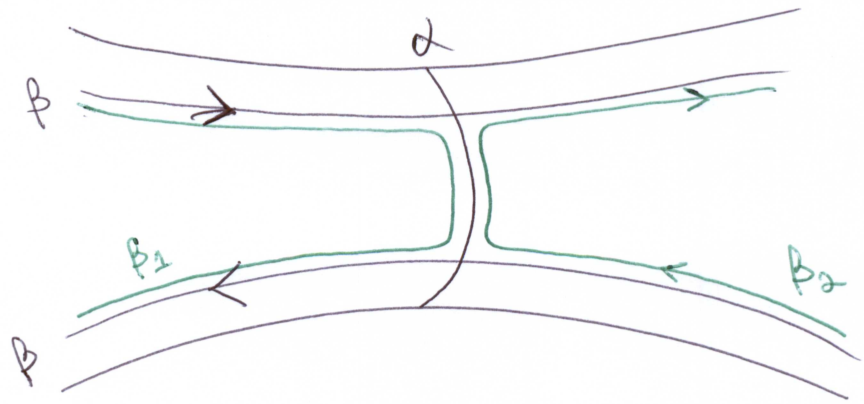
Then β' must cross β exactly once, from the left side to the right, so $i(\beta,\beta')=1 \Rightarrow$

$d(\beta,\beta')=2$ and β' is not isotopic into ∂M.

By induction $d(\alpha,\beta) \leq d(\alpha,\beta') + d(\beta',\beta)$

$$\leq 2(k-1)+1+2 = 2k+1.$$

Case II: Assume β can be oriented as shown:



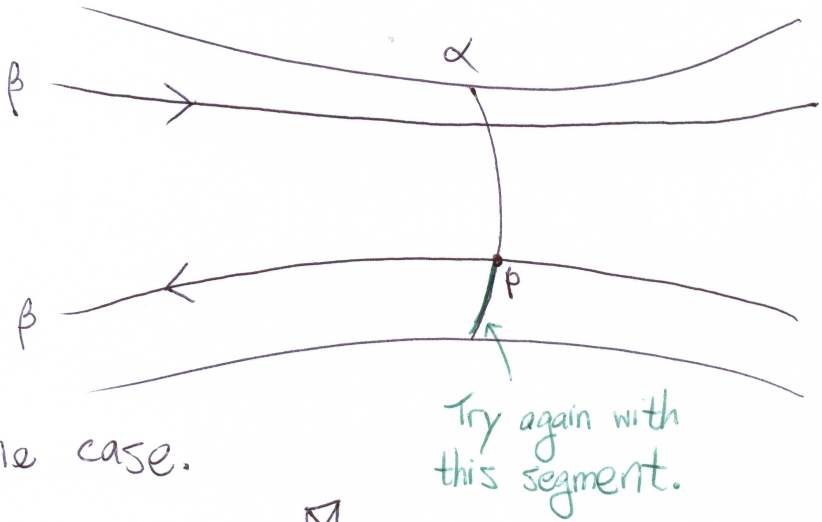
Then perform surgery to produce β_1 & β_2.

Each β_j is homot. nontrivial, & $i(\beta_j,\alpha) \leq k-2$.

However, $i(\beta,\beta_j)=0$ so we must show at least one of the β_j is not isotopic into ∂M.

If β_j is not isotopic into ∂M then we're done by induction. Suppose β_1 & β_2 are homotopic to components of ∂M. Then the component of $M-\beta$ containing the β_j must be a thrice-punctured sphere.

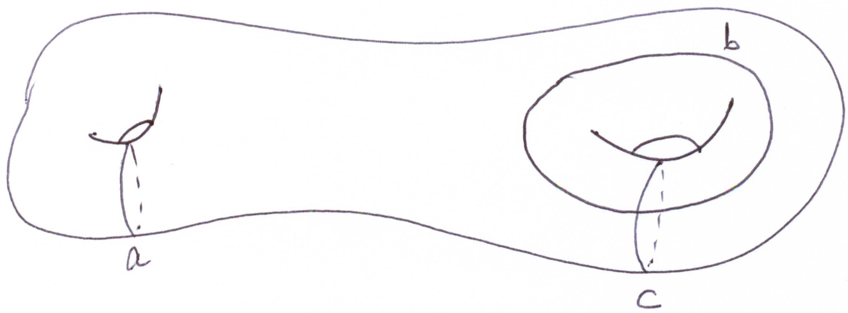
Apply the above argument to the other segment of $\{\alpha \text{ cut along } \alpha\cap\beta\}$ containing the point p as shown. If we end up again in this case then M must be a 4-times punctured sphere, which we assumed is not the case.



$\boxtimes$

Notice the is no reverse inequality; one cannot bound distance from below by intersection number.

E.g.



$$i(D_c^n(b), b) \xrightarrow{n\to\infty} \infty, \text{ but}$$

$$d(D_c^n(b), b) \leq d(D_c^n(b), a) + d(a,b) = 2.$$

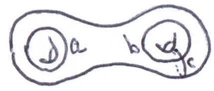
So it's <u>not</u> obvious that the curve graph has ∞ diameter.

Lecture 6

May 20, 2009

It's worth noting the geometric meaning of small distances in the curve graph:

$d(b,c) = 1 \iff$ b + c are disjoint

$d(b,c) = 2 \iff \exists$ a disjoint from b + c 

$d(b,c) > 2 \iff$ $M - (b \cup c)$ is a union of disks and annuli. (If $\partial M = \emptyset$ then $M - (b \cup c)$ is only disks. The annuli always have one ~~comp~~ boundary component lying in ∂M.)

Seeing the difference between distance 7 and 70 is not so easy.

The next goal is to prove that the curve complex has infinite diameter. More precisely, we'll show the curve graph has infinite diameter. The argument here is due to T. Kobayashi "Heights of simple loops and pseudo-Anosov homeomorphisms" (Prop 2.2). The argument requires several preliminary facts from Teichmüller theory. Let me state them first, then ~~prove the~~ give Kobayashi's argument. We'll go back and fill in the facts afterward, if we have time.

FACTS:

1. The space of projective measured foliations $\mathcal{PMF}$ is compact. (In fact it's a sphere, but we won't need this.)
2. The intersection number $i: \mathcal{S} \times \mathcal{S} \to \{0,1,2,\dots\}$ extends to a continuous fct. $i: \mathcal{MF} \times \mathcal{MF} \to [0,\infty)$.
3. Recall the definition of a pseudo-Anosov homeomorphism. We will need the existence of at least one pseudo-Anosov homeo with unstable ~~lamination~~ foliation $\mathcal{F}$. Moreover, we'll need the following fact that we have not seen at all previously:

$$i(\mu, \mathcal{F}) = 0 \Rightarrow \mu = \mathcal{F}.$$

Let $\mathcal{Y}_c$ be the measured foliation determined by $c \in \mathcal{S}$. From the def'n of pseudo-Anosov it follows that $\mathcal{Y}$ has no closed leaves. In particular, $i(\mathcal{Y}_c, \mathcal{Y}) > 0$, + $\mathcal{Y}_c \neq \mathcal{Y}$.

4. Recall there is an embedding $\mathcal{S} \hookrightarrow \mathbb{P}\mathcal{M}\mathcal{Y}$. We'll use that the image is dense.

Thm: The curve graph of M has infinite diameter.

Pf: Pick $c \in \mathcal{S}$ and let $Z_n \subset \mathbb{P}\mathcal{M}\mathcal{Y}$ be the (compact) closure of the set $\{b \in \mathcal{S} \mid d(b,c) \leq n\}$.

As above, let $\mathcal{Y}$ be the unstable foliation of some ψ-Anosov homeomorphism. ~~with~~ By the remarks in Fact 3 above, $Z_0 = \{c\} \neq \{\mathcal{Y}\}$. Suppose by induction that $Z_k \cap \{\mathcal{Y}\} = \emptyset$ $\forall$ $k<n$. In search of a contradiction assume $\mathcal{Y} \in Z_n$. Then $\exists$ $\{b_j\} \subset \mathcal{S}$ s.t. $d(b_j, c) = n$ and $b_j \to \mathcal{Y}$ in $\mathbb{P}\mathcal{M}\mathcal{Y}$. Also $\exists$ $\{a_j\} \subset \mathcal{S}$ s.t. $d(a_j, c) = n-1$ + $d(b_j, c_j) = 1$, implying $i(b_j, c_j) = 0$. Up to subsequence $a_j \to \lambda \in Z_{n-1}$. By continuity

$$0 = i(a_j, b_j) \to i(\lambda, \mathcal{Y}) \quad \Rightarrow \quad \lambda = \mathcal{Y}.$$

Contradiction. $\therefore$ $\forall$ n, $\mathcal{Y} \notin Z_n$.

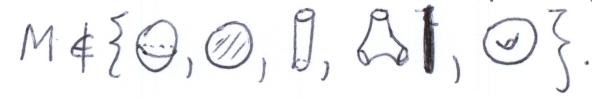
By Fact 4, $\exists$ scc $c_n \in \mathcal{S}$ s.t. $c_n \in \mathbb{P}\mathcal{M}\mathcal{Y} - Z_n$.

Then $d(c, c_n) > n$. $\boxtimes$

A overview of the general Masur-Minsky machinery

To state the general bounded geodesic image theorem we must define subsurface projections.

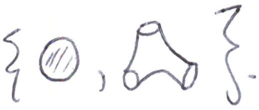
Suppose $X \subset M$ satisfies:

- M is a compact oriented connected surface with (possibly empty) boundary. ~~not~~ Assume $M \notin$ .

  (For $M =$  join scc's with intersection number 1.
  For $M =$  join scc's with intersection number 2.)

- X is a connected proper compact subsurface s.t. $X \hookrightarrow M$ induces an injection $\pi_1 X \hookrightarrow \pi_1 M$,

  X is not freely homotopic into ∂M,

  $X \notin$ . (X also is not )

  For simplicity, let's assume $X \neq$ . (This case is pesky.)

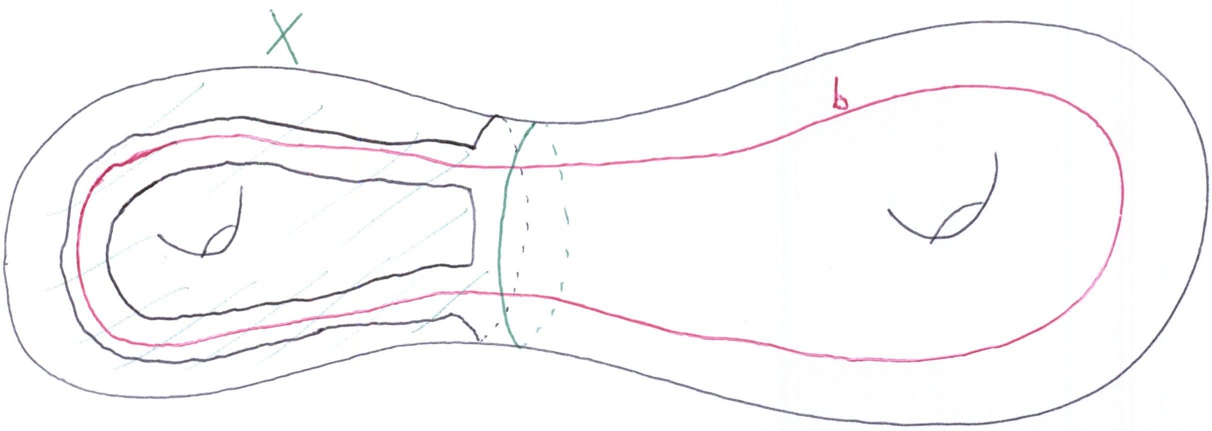
Define projection

$$\pi_X : \mathcal{C}(M) \longrightarrow \left(\begin{array}{c}\text{subsets of}\\ \mathcal{C}(X)\end{array}\right)$$

($\mathcal{C}$ indicates the curve graph.)

Pick $b \in \mathcal{C}(M)$. Assume we chose b to intersect ∂X minimally in its homotopy class. If $b \cap X = \emptyset$ then $\pi_X(b) = \emptyset$. If $b \subset X$ then $\pi_X(b)$ is simply $\{b\}$, thought of as a scc of X. Otherwise let $\{b_1, b_2, \ldots, b_n\}$ be the components of $b \cap X$. For each b_i let $X_i \subset \partial X$ be the components of ∂X intersecting b_i, and consider a small closed neighborhood

K of $b_i \cup X_i$. Let S_i be the components of ∂K that are homotopically nontrivial and not homotopic into ∂X. S_i must be nonempty. Let $\pi_X(b) = \bigcup S_i$. The diameter of $\pi_X(b)$ is ≤ 2.



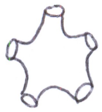
Note that some components of S_i may be homotopic in X, as in the picture's example.

[Bounded geodesic image thm (Masur-Minsky): Let $g \subset C(M)$ be a (possibly infinite) geodesic such that $\pi_X(v) \neq \emptyset$ for every vertex v of g. Then there is a constant $D(M)$ such that the set $\bigcup_{v \in g} \pi_X(v)$ has diameter at most D.

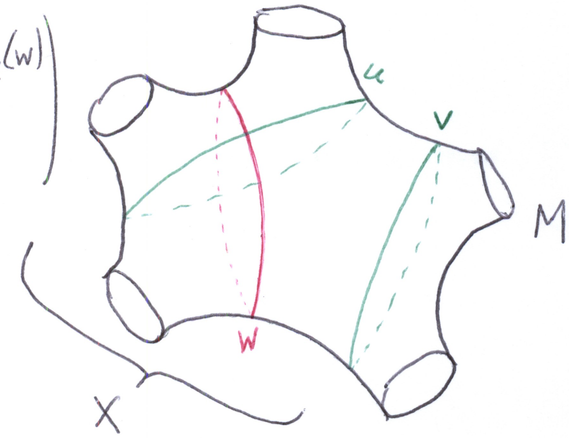
[Gromov hyperbolicity: $C(M)$ is Gromov hyperbolic.

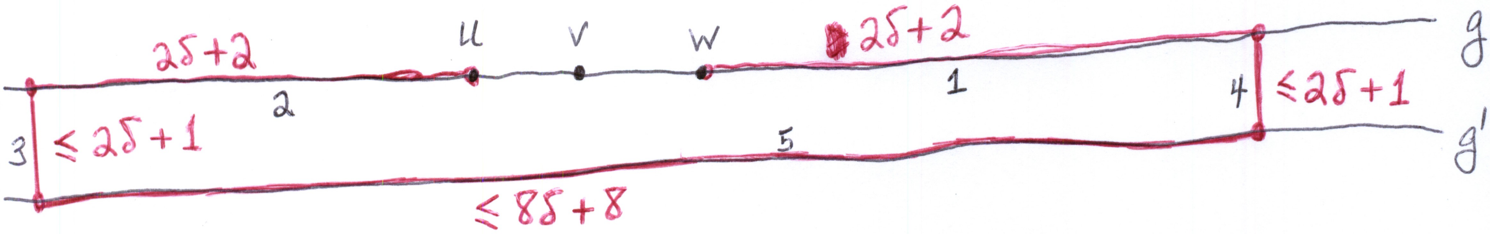
(I actually should write these in the other order.)

For now let's assume Gromov hyperbolicity with constant δ and try to get an intuition for the Bounded Geodesic Image Theorem. (This example is based on one from Masur-Minsky II.)

Suppose $M =$ . Let g be a (long) geodesic segment $\ldots, u, v, w, \ldots$. Let g' be another long geodesic segment with endpoints distance 0 or 1 from the endpoints of g. Then g & g' are $(2\delta + 1)$-fellow travelers.

(Note that $\pi_X(u)$ & $\pi_X(w)$ are distance 1 in this picture.)





Let $X \subset M$ be the closure of the 4-punctured sphere component of $M - v$. Then $u, w \subset X$.

Claim: If $\pi_X(u)$ & $\pi_X(w)$ are far apart in $\mathcal{C}(X)$ then g' must pass through v. (Not the case in the picture!)

Suppose g' does not pass through v. Form the red path in $\mathcal{C}(M)$ from u to w as shown:

1. go forward along g from w distance $2\delta + 2$
2. go backward along g from u distance $2\delta + 2$
3. skip over to g' with a path of length $\leq 2\delta + 1$ (by fellow traveler property)
4. skip over to g' with a path of length $\leq 2\delta + 1$

5. Close up with a path along g' of length $\leq 8\delta + 8$ (by the triangle inequality)

Every point of the red path is <u>not</u> v. So we can project it to $C(X)$.

Lemma: If $d_{C(M)}(b,c) = 1$ then $\operatorname{diam}(\pi_X(b \cup c)) \leq 2$.

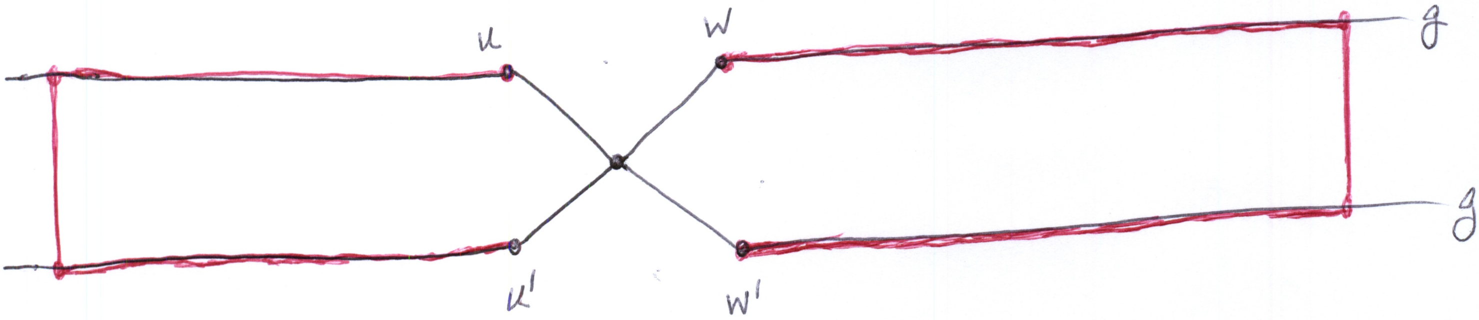
So π_X(red path) can be modified slightly to make a path of length at most $2 \cdot (2\delta + 2 + 2\delta + 2 + 2\delta + 1 + 2\delta + 1 + 8\delta + 8)$

$$= 32\delta + 28$$

We see that $d_{C(X)}(\pi_X(u), \pi_X(w)) > 32\delta + 28$

implies g' must pass through v.

Assume now that g' passes through v.



Let u' + w' be as in the picture. Using the red paths, and a similar argument, one can bound $d_{C(X)}(\pi_X(u), \pi_X(u'))$ and $d_{C(X)}(\pi_X(v), \pi_X(v'))$. It follows that ~~the~~ geodesics $[uw]$ and $[u'w']$ are fellow travellers. All this followed from knowing that the endpoints of g + g' are close to each other.

May 27, 2009

Geodesics in Teichmüller space and the curve complex

The next goal is to understand how to build (quasi-) geodesics in the curve complex. Given that the curve complex is not locally finite, it's not obvious how to find them.

To begin we must back up to Teichmüller space. Recall the def'n of Teichmüller space.

$$\mathcal{T}(M) = \left\{ (X, m) \;\middle|\; \begin{array}{l} X \text{ hyperbolic surface} \\ \text{w/ homeom. } m: M \to X \end{array} \right\} \Big/ \sim$$

where $(X, m_X) \sim (Y, m_Y) \iff \left(\begin{array}{l} \exists \text{ isometry } i: X \to Y \\ \text{s.t. } i \circ m_X \sim m_Y \end{array} \right)$.

Sadly, for this part of the story this is the wrong definition. We need the complex analytic definition

$$\mathcal{T}(M) = \left\{ (X, m) \;\middle|\; \begin{array}{l} X \text{ Riemann surface} \\ \text{w/ homeom. } m: M \to X \end{array} \right\} \Big/ \sim$$

where $(X, m_X) \sim (Y, m_Y) \iff \left(\begin{array}{l} \exists \text{ conformal map } i: X \to Y \\ \text{s.t. } i \circ m_X \sim m_Y \end{array} \right)$.

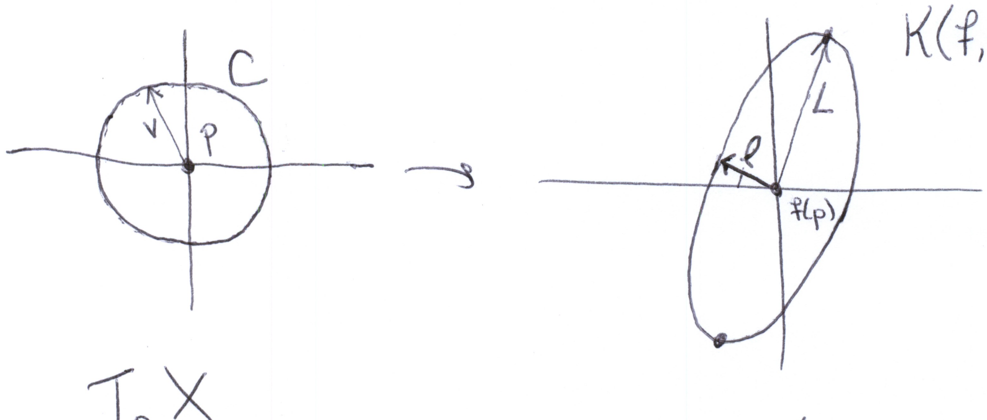
Next we need to know what a quasiconformal map is, at least in the differentiable case.

Sps $f: X \to Y$ is differentiable at $p \in X$. Using only structures on X & Y, we don't know how long vectors are in T_pX & $T_{f(p)}Y$. However, ratios $\frac{\|v_1\|}{\|v_2\|}$ are well defined for v_i in T_pX (or $T_{f(p)}Y$). So pick $v \in T_pX - 0$ and take the circle C of vectors $w \in T_pX$ s.t. $\frac{\|w\|}{\|v\|} = 1$.

Define the quasiconformal constant of f at p to be $\sup_{w_1, w_2 \in C} \frac{\|df_p w_1\|}{\|df_p w_2\|}$, & call it $K(f,p)$. Note $K(f,p) \geq 1$, & $=1$ iff df_p is conformal.

Let the quasiconformal constant of f be the smallest K s.t. $K(f,p) \leq K$ a.e. (We're ignoring many analytic details here, but the a.e. is important. We cannot reasonably require f be differentiable everywhere. That would rule out many or most interesting examples.)

$K(f,p) = L/\ell$.



T_pX $\qquad$ $T_{f(p)}Y$

The Teichmüller metric on $\mathcal{T}(M)$ is

$$d_{\mathcal{T}}((X,m_X),(Y,m_Y)) := \frac{1}{2}\log K$$

where $K = \inf \left\{ \begin{array}{l} \text{quasiconformal} \\ \text{constant of} \\ \text{homeo. } f \end{array} \middle| \begin{array}{l} f: X \to Y \text{ s.t.} \\ m_Y \sim f \circ m_X \end{array} \right\}.$

There is a not-so-obvious fact hiding under here: If $K=1$ then $d_{\mathcal{T}}=0$, so in fact X & Y are conformally equivalent. This means having a q.c. constant of 1 a.e. implies conformality.

Everyone gets to define a metric on $\mathcal{T}(M)$. Why is this one interesting? Because of Teichmüller's thm., which I'll try to explain next.

Fix $(X,m_X),(Y,m_Y) \in \mathcal{T}(M)$.

Part 1: $\exists!$ homeom. $f: X \to Y$ s.t. $m_Y \sim f \circ m_X$ and the q.c. constant of f <u>equals</u> $d_{\mathcal{T}}((X,m_X),(Y,m_Y))$. In other words the infimum of $d_{\mathcal{T}}$ is uniquely realized. f is called the Teichmüller map.

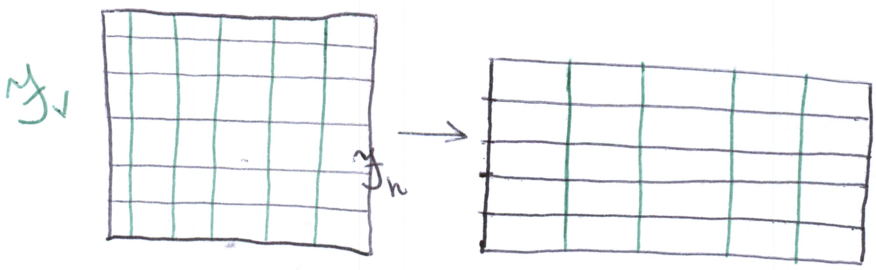
Part 2: We can describe the Teichmüller geodesic from (X,m_X) to (Y,m_Y) as (X_t, m_X) for a 1-parameter family of conformal structures X_t, $1 \le t \le e^{2d_{\mathcal{T}}(X,Y)}$,

Notice this is the qc-constant of f!

as follows. Let $X_1 = X$.

$\exists$ a pair of transverse measured singular foliations $\mathcal{F}_h$ and $\mathcal{F}_v$ on X

(think a horizontal foliation + a vertical foliation)
such that at a nonsingular pt of X the conformal structure of X_t is given by stretching by a factor of $\sqrt{t}$ in the horizontal direction and squishing for a factor of $\sqrt{t}$ in the vertical direction.



nonsingular pt.

This defines a path (X_t, m_X) in $\mathcal{T}(M)$ s.t.
$(X_{t_0}, m_X) = (Y, m_Y)$ for $t_0 = e^{2d_T(X,Y)}$

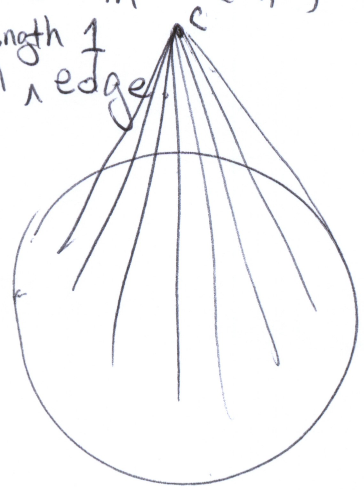
This means the Teichmüller map is <u>affine</u> away from the common singular pts. of $\mathcal{F}_h$ + $\mathcal{F}_v$.

Summary: The Teich. metric has unique geodesics with a fairly explicit model for how the surface is changing.

Mention the bi-infinite Teich. geodesic associated to a pseudo-Anosov homeom..

Recall the notion of a Margulis constant μ for hyperbolic surfaces. We need the fact that, on any hyperbolic surface, two closed curves of length $< \mu$ never intersect.

Next I'll define the electric Teich. space $\mathcal{T}_{el}(M)$. Start with $\mathcal{T}(M)$ equipped with the Teich. metric. Now for each scc $c \in \mathcal{S}(M)$, add a disjoint point c to $\mathcal{T}(M)$. Finally, if the length of c in (X, m) is less than or $=$ to μ then join (X, m) & c by an length 1 edge.



This makes any pair of surfaces where c is short distance ≤ 2 in $\mathcal{T}_{el}$. There is an obvious map $\mathcal{C} \longrightarrow \mathcal{T}_{el}$ taking a curve c to the added point $c \in \mathcal{T}_{el}$. ~~This map is clearly~~

FACT: For any set of pairwise ~~dis~~ disjoint curves $\{c_i\} \subset \mathcal{S}$ on M, $\exists$ $(Y, m_Y) \in \mathcal{T}(M)$ where all the curves c_i are short, i.e. length $\leq \mu$.

Corollary: $\mathcal{C} \longrightarrow \mathcal{T}_{el}$ is 2-Lipschitz.

FACT: ~~Th~~ $\exists$ D s.t. the image of $\mathcal{C} \to \mathcal{T}_{el}$ is D-dense. I.e. there is always a curve of medium length, and it can be shortened without moving too far in $\mathcal{T}(M)$.

Next define $\Phi : \mathcal{Y} \longrightarrow \left\{ \begin{matrix} \text{subsets of} \\ \mathcal{C} \end{matrix} \right\}$

$$(X, m_X) \longmapsto \left\{ \begin{matrix} \text{the set of shortest} \\ \text{curves on } X \end{matrix} \right\}$$

FACT: $\exists$ $b > 0$ s.t. if $d_{\mathcal{Y}}((X, m_X), (Y, m_Y)) \leq 1$ then

$$\operatorname{diam}(\Phi(X) \cup \Phi(Y)) \leq b.$$

This implies Φ is Lipschitz. We'd like to electrify Φ. This is no problem, just define $\Phi_{el}(c) := c$ for the added points. The resulting map $\Phi_{el} : \mathcal{Y}_{el} \longrightarrow \mathcal{C}$ is Lipschitz, and it is a quasi-inverse to the map $\mathcal{C} \longrightarrow \mathcal{Y}_{el}$.

We conclude that $\mathcal{C} \longrightarrow \mathcal{Y}_{el}$ is a quasi-isometry.

Thm (Masur - Minsky): The map ~~$\mathcal{Y}_{el} \to \mathcal{C}$~~ $\Phi : \mathcal{Y} \longrightarrow \mathcal{C}$ sends Teichmüller geodesics to quasigeodesics of $\mathcal{C}$ with uniform quasigeodesic constants.

This is our desired model of geodesics in $\mathcal{C}(M)$, as images under Φ of Teichmüller geodesics.

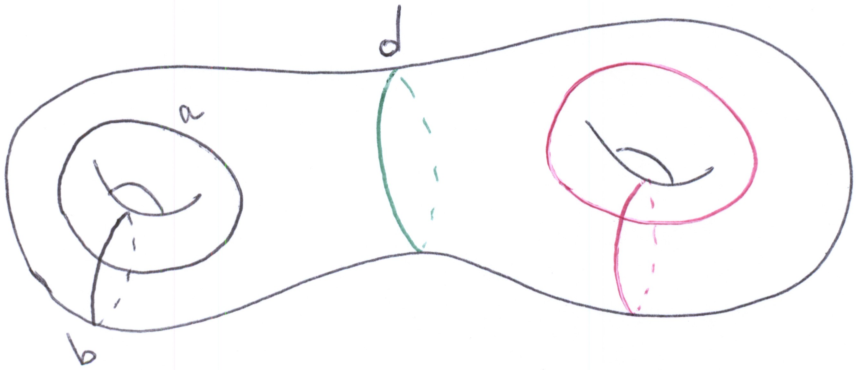
In a related vein, let's finish with a brief introduction to tight geodesics.

Define a geodesic in $\mathcal{C}(M)$ to be a sequence $\Sigma_0, \Sigma_1, \dots, \Sigma_n$ of <u>simplices</u> of $\mathcal{C}(M)$ s.t.:

$\forall$ i, j and any curves $c_i \in \Sigma_i$, $c_j \in \Sigma_j$,

$$d(c_i, c_j) = |i-j|.$$

Notice there is a lot of "local" ambiguity in a geodesic of $\mathcal{C}(M)$. For example, let's return to a common example



$d(a,b) = 2$, but if c is <u>any</u> curve in the right half of the surface (3 are shown) then $\{a, c, b\}$ is a geodesic. To eliminate this ambiguity, Masur-Minsky introduced the notion of a <u>tight</u> geodesic. For any pair Σ_i & Σ_{i+2} there is a minimal connected π_1-injective subsurface $R \subset M$ containing all the curves of Σ_i & Σ_{i+2}. As $d(\Sigma_i, \Sigma_{i+2}) = 2$, we know $R \neq M$.

Finally, our geodesic is <u>tight</u> if every curve of Σ_{i+1} is homotopic into ∂R.

For example, in the above picture the only tight geodesic is $\{a, d, b\}$. Notice tightness is local.

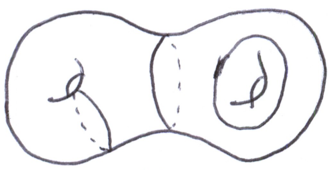
Thm (MM): Between any two points of $\mathcal{C}$ $\exists$ a finite number of tight geodesics.

Laminations, Train tracks, & singular foliations June 10, 2009

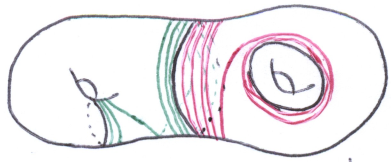
Fix M a closed hyperbolic surface. (To keep things simple today we'll assume M has no boundary.)

Def: A geodesic lamination λ on M is a union of disjoint geodesics of M forming a closed set. (Note that a geodesic is either closed or bi-infinite.)

E.g:



a multicurve



a multicurve with additional "spiraling" geodesics

Each component of $M-\lambda$ has area $\geq \pi$, because it has geodesic boundary & one can apply Gauss-Bonnet.

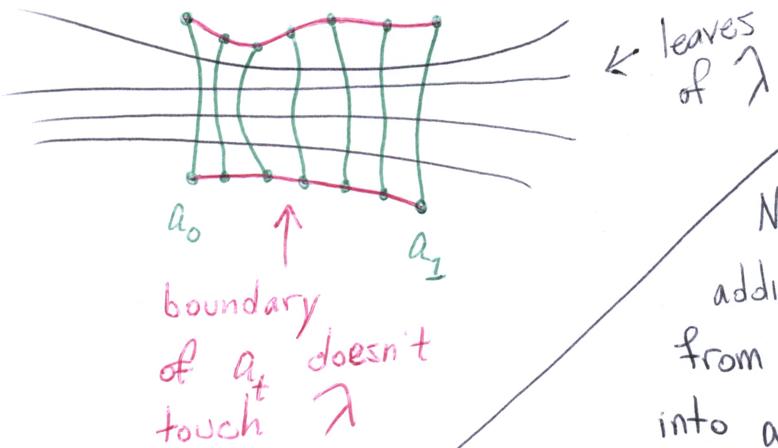
$\Rightarrow$ $M-\lambda$ has $\leq \frac{\text{Area}(M)}{\pi} = 2 \cdot |\chi(M)|$ components.

With a little more work one can show λ has area 0 in M. (For this and much more, see Chapter 8 section 5 of Thurston's Notes.)

With a homeomorphism $f: M \to N$, for N a hyperbolic surface, we can push λ to N by identifying a geodesic in M with a distinct pair in $\partial\tilde{M}$, & then using the π_1-equivariant homeom $\partial\tilde{M} \to \partial\tilde{N}$. So the choice of metric on M is just for convenience.

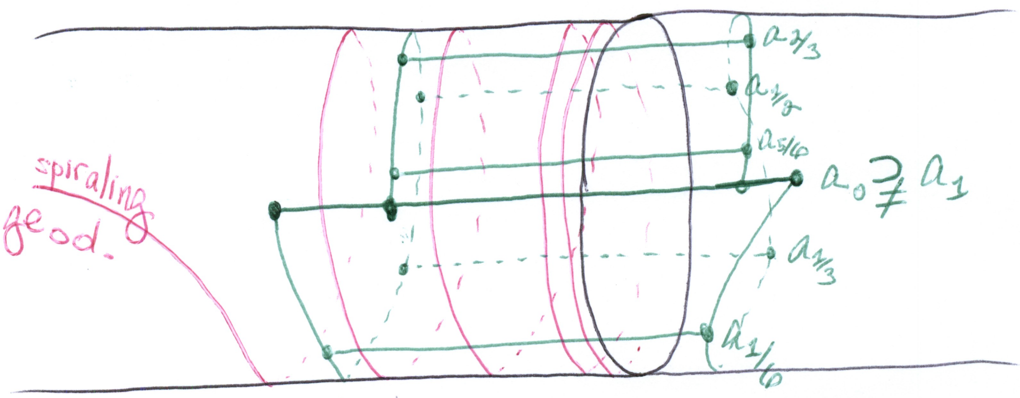
Def: A measured geodesic lamination on M is a geodesic lamination λ together with a measure μ on the set of compact arcs of M transverse to λ satisfying:

1. $\mu(a) < \infty$ for any compact arc a transverse to λ
2. If a_t is a 1-parameter family of compact arcs transverse to λ s.t. $\partial a_t \cap \lambda = \emptyset$ for all t then $\mu(a_0) = \mu(a_t)$ $\forall$ t.



3. We assume μ has full support, i.e. $a \cap \lambda \neq \emptyset \Rightarrow \mu(a) > 0$.

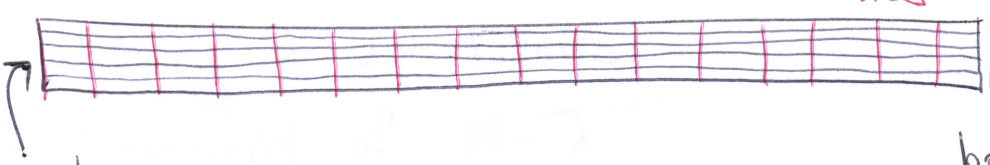
Note the multicurve with additional "spiraling" geodesics from page 1 cannot be made into a measured geodesic lamination. A measured geod. lam'n cannot have an infinite geod. spiraling into a closed geod.



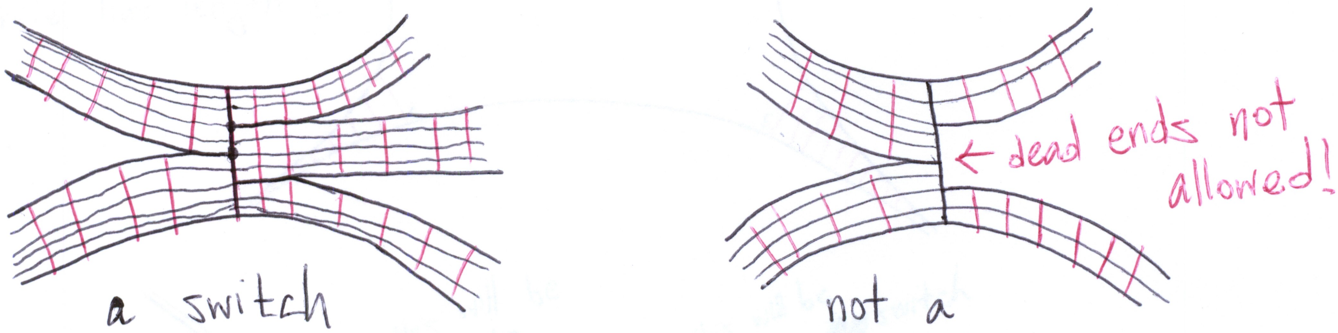
closed geodesic of a lamination

One could, in this situation, find a family a_t of transverse arcs s.t. $a_1 \subsetneq a_0 \Rightarrow \mu(a_t) = 0$.

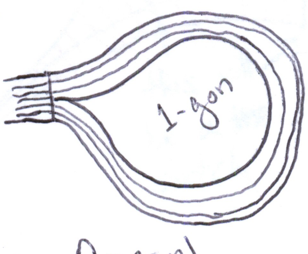
The next goal is to define train tracks. It's best to draw lots of pictures. A ~~piece~~ branch of track is an embedded square in M with its vertical + horizontal foliation. The horizontal foliation forms the leaves of the ~~piece~~ branch. The vertical foliation form the ties.



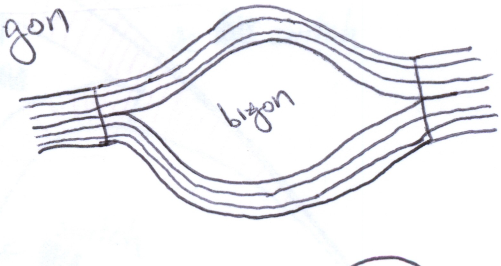
A switch is a union of ~~pieces~~ branches glued along boundary ties so there are no dead-end leaves.



A train track τ on M is a collection of ~~pieces~~ branches and switches so there are no dead-end leaves, and no component of $M-\tau$ is diffeom. to a 1-gon e.g.



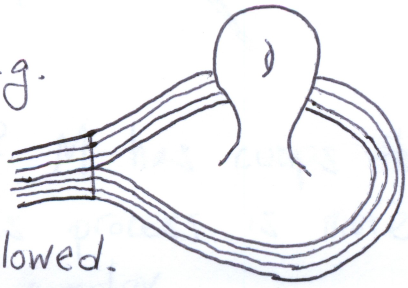
or a bigon



.

(or a 0-gon)

Note that topology in $M-\tau$ is allowed. E.g.



is allowed.

~~Note~~ Now assume λ is a geodesic lamination with measure μ. Note that inside any fixed ~~plane~~ branch of τ the measure of a tie is constant. Moreover, ~~the~~ at a switch the total measure of the ties on the left equals the total measure of the ties on the right. This motivates the def'n

Def: A weighted train track assigns a positive weight to each branch such that at each switch the total weights on each side are equal.

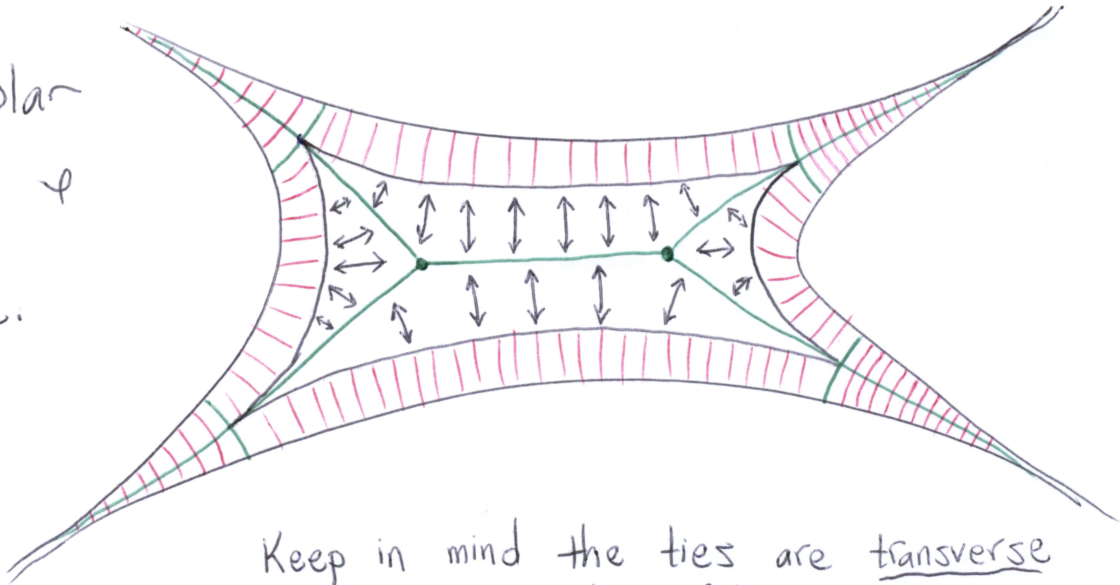
So our construction builds a weighted train track from a measured geodesic lamination.

Notice that by choosing ε smaller we obtain finer approximations of our lamination.

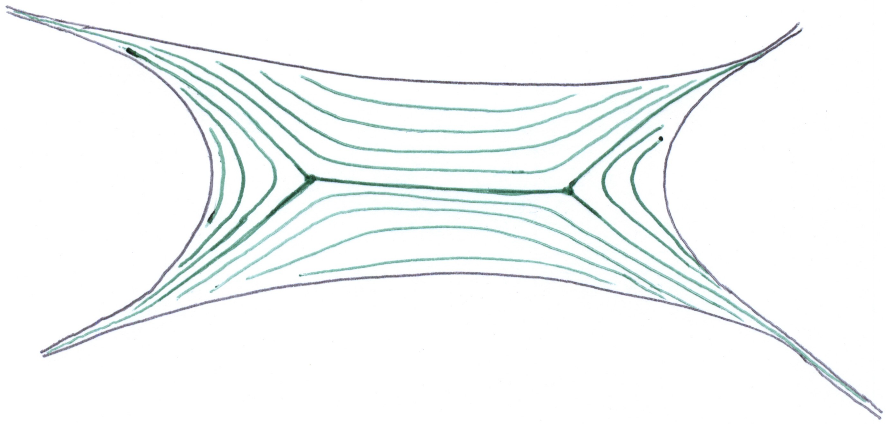
Building a measured singular foliation from a weighted train track is easy, if a bit technical to nail down. ~~Simply fill in the complementary regions collapse~~

First add some singular leaves to the complement of τ and then collapse the rest of the complement of τ onto the singular leaves.

Add singular leaves & then collapse.



Keep in mind the ties are <u>transverse</u> to the resulting foliation.



There is ambiguity when choosing how to add singular leaves. All choices are Whitehead equivalent. When a complementary region has some topology then adding singular leaves is slightly more complex. We'll skip these details here. This gives a singular foliation. What about the measure? For each branch of the train track of weight w put a uniform Lebesgue measure on the ties of total measure w. This measure transfers in the obvious way to curves transverse to the singular foliation.

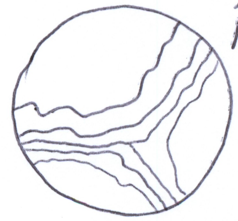
This describes maps

$$\left\{\begin{matrix}\text{measured}\\ \text{laminations}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{weighted}\\ \text{train}\\ \text{tracks}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{measured}\\ \text{singular}\\ \text{foliations}\end{matrix}\right\}.$$

We'll complete the picture with a map

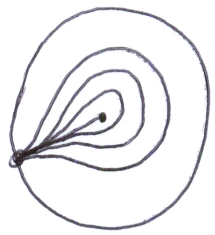
$$\left\{\begin{array}{l}\text{measured}\\ \text{singular}\\ \text{foliations}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{measured}\\ \text{laminations}\end{array}\right\}.$$

Consider a measured singular foliation $\mathcal{F}$ on M. Lift $\mathcal{F}$ to a measured singular foliation $\tilde{\mathcal{F}}$ on $\tilde{M}$. The boundary at infinity $\partial_\infty \tilde{M}$ is S^1. Each smooth leaf of $\tilde{\mathcal{F}}$ lifts to a curve in $\tilde{M}$ with endpoints in $\partial_\infty \tilde{M}$.



Claim: The endpoints of a smooth leaf cannot coincide.

Pf:



If so there must be a "dead end" singular leaf, as shown. This is not allowed. $\boxtimes$

So we can pull each smooth leaf in $\tilde{M}$ tight to a geodesic with the same endpoints.

FACT: Distinct leaves pull tight to disjoint geodesics.

This defines a π_1-equivariant map $\text{tight}: \text{smooth leaves} \to \text{geods in } \tilde{M}$.

The image of $\text{tight}(\tilde{\mathcal{F}})$ is a π_1-invariant geodesic lamination λ of $\tilde{M}$. For arc a transverse to λ define the measure $\mu(a)$ as the measure of

$\text{tight}^{-1}(a \cap \lambda)$. This defines a π_1-invariant measured lamination on $\tilde{M}$ that descends to a measured lamination on M.