Lecture 1 at HUJI 2009-03-18
(Notes from a course on mapping class groups by Peter Storm)

Fix a compact oriented surface $M$ without boundary. Assume all maps preserve orientation unless explicitly stated otherwise. Assume $M$ is not a sphere.

Def: The group of $M$ is $\text{Mod}(M) = \pi_0(\text{Homeo}(M))$.

We will call elements of $\text{Mod}(M)$ mapping classes (orientation preserving homeos).

Thm (Dehn - Nielsen - Baer): The natural map

$$\text{Homeo}(M) \to \text{Out}(\pi_1 M)$$

defines a homomorphism $\text{Mod}(M) \to \text{Out}(\pi_1 M)$. This homomorphism is injective with image of index 2. The image is exactly orientation preserving maps of $\text{Out}(\pi_1 M)$, i.e. those acting trivially on $H^1(\pi_1 M; \mathbb{Z}) \cong \mathbb{Z}$.

E.g. if $M$ is a torus then $\text{Mod}(M) \cong \text{SL}_2 \mathbb{Z}$.

An example of a mapping class is the finite order homeomorphism:

\[ \text{rotate by } \frac{\pi}{2} \]

Another (more important) example is given by the Dehn twist. Let $c \subset M$ be an (embedded) homotopically nontrivial simple closed curve. A right hand Dehn twist about $c$, $D_c$, is a homeomorphism $M \to M$ with support in a small annular neighborhood of $c$. 
Put "coordinates" \((t, \theta)\) on an annular neigh. of \(c\) s.t. 
\[t \in (-\varepsilon, \varepsilon), \quad \theta \in S^1, \text{ and } c = \tilde{c}(0, \theta) \mid \theta \in S^1\].

Then define a homeomorphism \(f: M \to M\) with support in this annular model by 
\[
f(t, \theta) = \begin{cases} 
(t, e^{\frac{t^2}{2}}) & \text{for } t \leq 0 \\
(t, \theta) & \text{for } t > 0 
\end{cases}
\]

Then \(D_c\) is the mapping class of \(f\). \(D_c\) is also called the left-handed Dehn twist about \(c\). (We chose the left-handed orientation to follow N. Ivanov.) Note the definition of \(f\) involved an orientation of \(c\), namely which is the left side of \(c\), but \(D_c\) is independent of this choice. Obviously, we could have chosen \(f\) to be smooth with slightly more work.

On a torus, Dehn twists take a particularly simple linear form. Under the Dehn-Nielsen-Baer canonical homomorphism \(D_{\beta}\) corresponds to 
\[
\begin{pmatrix} \beta \rightarrow \beta \\
\alpha \rightarrow \alpha + \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\
1 & 1 \end{pmatrix}
\]

and \(D_{\alpha}\) corresponds to 
\[
\begin{pmatrix} \alpha \rightarrow \alpha \\
\beta \rightarrow \beta - \alpha \end{pmatrix} = \begin{pmatrix} 1 & -1 \\
0 & 1 \end{pmatrix}
\].
Here Dehn twists correspond to unipotent elements. Intuitively, Dehn twists are similar to unipotents in higher genera.

In lattices, unipotents are not "typical" elements. This is true here also. Without defining "typical" precisely, I claim Dehn twists are not the typical elements of $\text{Mod}(M)$. What is an example of a typical element? Consider $c = D_{2} D_{3}$, which corresponds to $(2 \frac{1}{3})$, a hyperbolic element of $SL_{2}\mathbb{Z}$.

I claim $c$ is typical. In particular, $c$ has the following property:

**Def.:** Let $\mathcal{A}$ be the set of isotopy classes of homologically nontrivial simple closed curves on $M$.

Clearly $\text{Mod}(M) \cap \mathcal{A}$.

Then for all $k \neq 0$, $c^{k}$ acts on $\mathcal{A}$ without a fixed pts., i.e. if $c$ is a nontrivial s.c.c. then $c^{k}(c)$ is not homotopic to $c$.

We'll call this property (§). In this case it is easy to prove using linear algebra. For $(0, k)$ is $\frac{k}{2}(m,n)$.

How can be build higher genus mapping classes with property (§)?

**Remark:** Suppose $\alpha$ and $\beta$ are simple closed curves on $M$ satisfying

- they are homologically nontrivial
- $M - (\alpha \cup \beta)$ is a set of disks
- $\#\alpha \cap \beta$ and $\#\alpha' \cap \beta'$ then $\#|\alpha \cap \beta| \leq \#|\alpha' \cap \beta'|$, i.e. $\alpha \cap \beta$ intersect minimally.
For example if $M$ has genus 2 then one could choose $\alpha \neq \beta$ as:

Then I claim that $D_{\alpha}^2 D_{\beta}$ has property (x) and should be considered a typical element of $\text{Mod}(M)$. See (FLP, Ex. 13).

Measured singular foliations

A measured singular foliation $\mathcal{F}$ is:

- a finite set $S = \{p_1, p_2, \ldots, p_N\} \subset M$ (the "singular set")
- an atlas $\mathcal{U}_x, \phi_x^2$ on $M - S$ such that each $U_x = (a_1, b_1) \times (a_2, b_2)$

Each transition $\phi_x^{-1} \circ \phi_y$ takes horizontal lines to horizontal lines, and each transition for $\phi_x^{-1} \circ \phi_y$ locally preserves vertical distances. More specifically, these conditions say:

$$(\phi_x^{-1} \circ \phi_y)(x_i, y_i) = (x'_i, y'_i) \text{ for } i \in \{1, 2 \} \text{ then}$$

$$|y_i - y'_i| = |y'_i - y'_i|.$$  

This atlas defines a horizontal foliation on $M - S$.  

Each singular point $p$ has a neighborhood such that the horizontal foliation looks like one of

\[ \text{\ldots (higher order singularities)} \]

\[ Y \] defines a (positive) measure on the set of arcs transverse to the horizontal foliation. Namely, if such an arc $p: [0,1] \to M-S$ is contained in a single chart then

\[ Y(p) = |(y\text{-coord. of } p(1)) - (y\text{-coord of } p(0))|. \]

In general, cut the arc into small pieces and sum.

Note that if $X(M) < 0$ then the singular set $\Sigma$ must be nonempty, with $\Sigma + \Sigma'$.

We will say two foliations are isotopic if there is a diffeomorphism $f$ on $M$ isotopic to the identity such that:

\[ f(S) = f(S') \]

the singular set of $Y'$

\[ f \text{ sends the horiz. foliation of } Y \text{ to that of } Y'. \]

\[ \text{If } p \text{ is an arc in } M-S \text{ transverse to } Y \text{ then } \]

\[ Y'(f(p)) = Y(p). \]

E.g. One could decompose the charts with a small diffeo of $M$.  

Next we'll describe Whitehead moves. Suppose \( l \) is a leaf of \( \mathcal{F} \) running from one singular leaf to another as shown.

In a neighborhood of \( l \), cut out an open diamond-shaped region as shown, and glue the endpoints of horizion leaves together. This produces the measured singular foliation on the right. This process is reversible. Both are called Whitehead moves. The two measured singular foliations are Whitehead equivalent.

Let \( A \) be the set of measured singular foliations on \( M \).

Define

\[
\mathcal{F}_1 \sim \mathcal{F}_2 \iff \begin{cases} \exists \text{ a sequence of Whitehead moves on } \mathcal{F}_1 \text{ producing a measured sing. foliation isotopic to } \mathcal{F}_2 \\ \end{cases}
\]

and

\[
\mathcal{F}_1 \sim_{\nu} \mathcal{F}_2 \iff \begin{cases} \exists \mathcal{F}_3 \text{ s.t. } \mathcal{F}_2 \sim \mathcal{F}_1 \sim \mathcal{F}_3 \text{ and the measure on } \mathcal{F}_3 \text{ is a constant multiple} \\ \text{of the measure on } \mathcal{F}_2 \\ \end{cases}
\]

Then

\[
\mathcal{M}(\mathcal{F}) \equiv \frac{A}{\nu_1} \text{ and } \text{PM}(\mathcal{F}) := \frac{A}{\nu_2}.
\]
Def: $\mathcal{Y}^u + \mathcal{Y}^s$ foliations on $M$ are transverse if they have the same singular sets and are transverse on $M-S$.

Def: $f : M \to M$ a homeomorphism is pseudo-Anosov (P-A) if $\exists \mathcal{Y}^u + \mathcal{Y}^s$ transverse singular measured foliations and $\lambda > 1$ s.t.

- $f$ sends leaves of $\mathcal{Y}^s$ to leaves of $\mathcal{Y}^s$
- $f(\mathcal{Y}^s) = \frac{\lambda}{\mu} \mathcal{Y}^s$ and $f(\mathcal{Y}^u) = \lambda \mathcal{Y}^u$

where these are equalities of measures.

Def: A multicurve $cM$ is an embedded 1-manifold (i.e. a finite union of simple closed curves) s.t. each component is homotopically nontrivial, and no pair of components are isotopic.

Thurston's classification of surface homeos: A homeomorphism $f: M \to M$ is isotopic to a homeo $g$ satisfying at least one of the following:

(i) $g$ has finite order in $\text{Homeo}(M)$

(ii) $\exists$ multicurve $cM$ s.t. $g(c) = c$

(iii) $g$ is pseudo-Anosov.

(If (iii) then not (i) and not (ii).)
Recall the def'n of Teichmüller space. Fix a closed oriented surface $M$ of genus $g > 1$. (We'll draw $M$ with genus $2$.) Define the set of pairs $(X, m)$ where $X$ is a hyperbolic surface and $m : M \to X$ is a homeomorphism. ($m$ stands for "marking.") Define the equivalence relation $\sim$:

$$(X, m_x) \sim (Y, m_y) \iff \exists \text{ isometry } f : X \to Y \text{ s.t. } f \circ m_x \text{ is homotopic to } m_y.$$ 

Then $\widehat{Y(M)} := \{ (X, m) \}_{\sim}$. For far,

this is only a set. Define a metric

$$d((X, m_x), (Y, m_y)) := \inf \{ \log K \mid \exists K\text{-bilipschitz homeomorphism } f : X \to Y \text{ s.t. } m_y \sim f \circ m_x \text{ homotopic} \}$$

on $\widehat{Y(M)}$, turning it into a metric space. (This metric is not particularly interesting, but it easy to define. All the well-known metrics on $\widehat{Y(M)}$ define the same topology.)

**Fact:** $\widehat{Y(M)} \cong \mathbb{R}^{3g-6}$ (due to maybe Teichmüller or Bers?)

**Mod(M) \cap \widehat{Y(M)}** by isometries via:

$$\varphi : (X, m_x) := (X, m_x \circ \varphi^{-1}).$$

Note this is a mapping class while this is a homeo. Check this is well defined!
How can we build a hyperbolic surface? Begin with some planar (hyperbolic) geometry. \exists right-angled hexagons in \( \mathbb{H}^2 \), e.g.

Prop: Label the edges of a hexagon as shown:

\[ \begin{array}{c}
\text{Proposition:} \quad l_a, l_b, l_c > 0. \quad \exists!
\end{array} \]

right-angled hyperbolic metric on the hexagon such that edge \( x \) has length \( l_x \) (for \( x \in \{a,b,c\} \)).

Proof sketch: (This proof is "Thurston-esque"). Wolog \( a > b > c \).

(A trig proof is possible. See Ratcliffe, Thm 3.5.14.) Fix \( a \) as a vertical geodesic in \( \mathbb{H}^2 \).

Imagine \( b + c \) swinging along geodesics as shown. Let \( x \) be the distance from \( a \) to \( b \).

Given \( x + \gamma \), let \( \gamma \) be the geodesic from \( b \) to \( c \). Label the angles of \( \gamma \) as shown. The goal is to find \( x + \gamma \)

s.t. \( \alpha = \beta = \frac{\pi}{2} \)

\[ \begin{array}{c}
\text{Diagram 1: }
\end{array} \]
Examine the behavior of $\alpha + \beta$ when at least one of $x,y$ is very large. This is shown in the picture. By continuity, $\exists$ values of $x,y$ producing $\alpha = \beta = \frac{\pi}{2}$. 

(Note: This proof sketch does not discuss uniqueness.)

Given two isometric right-angled hyperbolic hexagons, glue along the red edges to produce hyperbolic pants.

Similarly, given hyperbolic pants with geodesic boundary, add the red geodesics and cut to obtain right-angled hyperbolic hexagons.

Cor: $l_1, l_2, l_3$, $E!$ (marked) pants with 3 curves of length $l_1, l_2, l_3$.

Given two pants with boundary lengths as shown, we can glue to obtain a genus 2 hyperbolic surface. Similar constructions work in higher genus.

Note: there is ambiguity in the gluing.
To explicate the ambiguity, add an oriented point to each boundary circle. Then $a, b, c$ in the closed surface each have a well-defined twist $\alpha \beta \gamma \in [0, 2\pi)$. Intuitively this indicates that we must specify 6 real parameters $a, b, c, x, \alpha, \beta, \gamma$ to build a hyperbolic genus-two surface, suggesting the Teichmüller space $Y(\infty)$ should have dim $6$. This is correct, but not a proof. A similar construction in higher genera shows that $Y(M)$ should have dim $6g - 6$, and this count is correct.

Next I'll describe the Fenchel-Nielsen coord. system on $Y(M)$. For simplicity I'll describe it when $M$ has genus 2. Let's put the following nice hyper. metric on $M$.

These points are mirror images. Red curves are geodesics.
As before, for any \( l_1, l_2, l_3 \) and \( \theta_1, \theta_2, \theta_3 \) near 0 & can build a hyperbolic genus two surface \( X(l_1, l_2, l_3, \theta_1, \theta_2, \theta_3) \)

- \( l_1 \) is the length of the left curve,
- \( l_2 \) "middle" curve,
- \( l_3 \) "right" curve.
- \( \theta_1 \) is the twist of the left curve,
- \( \theta_2 \) "middle" curve,
- \( \theta_3 \) "right" curve.

Specify a marking \( m: M \to X(\vec{\ell}, \vec{\theta}) \) by defining generators

\[ \pi_1 M : \]

The basepoint is to the left of \( b \).

For small \( \theta_1 \), mark \( X(\vec{\ell}, \vec{\theta}) \) by \( g_1 \) taking \( g_i \) to \( g_i + h_i \) to \( h_i \). How to extend the for large \( \theta_1 \)? We adjust the marking. Specifically, if \( \theta_1 \in \mathbb{R} \) build a hyperbolic surface \( X(\vec{\ell}, \vec{\theta}) \) using twist parameters \( (\theta_1, \text{mod} 2\pi) \) and define a marking taking

\[
\begin{align*}
(g_1 \text{ on } M) & \mapsto (g_1 \text{ on } X(\vec{\ell}, \vec{\theta})) \\
(h_2 \text{ on } M) & \mapsto (g_1 h_2 \text{ on } X(\vec{\ell}, \vec{\theta})) \text{ where } k \text{ is the integer floor of } \frac{\theta_1}{2\pi}, \text{ i.e. the greatest int. } \leq \frac{\theta_1}{2\pi}, \text{ i.e. } \left\lfloor \frac{\theta_1}{2\pi} \right\rfloor.
\end{align*}
\]
\[(g_2 \text{ on } M) \mapsto (g_2^{-k_2} g_2 \cdot g_2^k \text{ on } X(l, \delta) \text{ where } k_2 = \lceil \frac{\theta_2}{2\pi} \rceil)\]

\[(h_2 \text{ on } M) \mapsto (g_2^{-k_2} g_2^{-k_3} h_2 \cdot g_2^k \text{ on } X(l, \delta) \text{ where } k_3 = \lceil \frac{\theta_3}{2\pi} \rceil)\]

and \(X = g_2 h_2 g_2^{-1} h_2 \) is the curve

Abusing notation slightly, let \(g_2 \) also denote the s.c.c. in the free homotopy class of \(g_2\). Then the marking \(\phi\) is better described as a homotopy equivalence with the following action on homotopy classes of s.c.c.'s.

\[
\begin{align*}
(g_1 \text{ on } M) & \mapsto (g_1 \text{ on } X(l, \delta)) \\
(h_1 \text{ on } M) & \mapsto (D_{g_1}^{k_1} h_1 \text{ for } k_1 = \lceil \frac{\theta_1}{2\pi} \rceil) \\
(g_2 \text{ on } M) & \mapsto (D_{g_2}^{k_2} h_2 \text{ for } k_2 = \lceil \frac{\theta_2}{2\pi} \rceil) \\
(h_2 \text{ on } M) & \mapsto (D_{g_2}^{k_2} D_{g_2}^{k_3} h_2 \text{ for } k_3 = \lceil \frac{\theta_3}{2\pi} \rceil)
\end{align*}
\]

Recall \(D_{g_2}\) is always a right-hand twist along \(*\), regardless of any orientation \(*\) may have.

This defines a marking \(m_{X(l, \delta)}\): just apply Dehn twists as prescribed in (4) to the original identity marking

\[m_{X(a,b,0,0,0)}: M \rightarrow X(a,b,0,0,0) \]

so for all \(l_1, l_2, l_3 > 0 \) and \(\theta_1, \theta_2, \theta_3 \in \mathbb{R}\) we have defined
a point \((X(\Hat{\ell}, \Hat{\theta}), m_{X(\Hat{\ell}, \Hat{\theta})})\) in \(Y(M)\).

This defines a set map

\[
\text{FN: } (0, \infty)^3 \times \mathbb{R}^3 \longrightarrow Y(M)
\]

Thm (Fenchel-Nielsen): This map is a homeomorphism.

More generally, in higher genus we can define

\[
\text{FN: } (0, \infty)^{3g-3} \times \mathbb{R}^{3g-3} \longrightarrow Y(M)
\]

- This map is always a homeomorphism.

(Generically, for any \(\phi \in \text{Mod}(M), (\text{FN}^{-1} \circ \phi \circ \text{FN})\) is a real-analytic diffeomorphism.)

We won't pursue this further.

This gives an explicit mental picture of \(Y(M)\) in terms of "length-twist" coordinates. Let's see an example using the above notation. We look at the left side of our surface as \(\theta_2\) increases past \(2\pi\). Fix \(l_1, l_2, l_3\) and \(\theta_2 = \theta_3 = 0\).

\[
\begin{align*}
\theta_2 &= 0 \\
\theta_2 < 2\pi & \quad \text{near } 2\pi \\
\theta_2 &= 2\pi, \text{ change the marking by } D_{\theta_2} \\
4\pi & \quad \theta_2 > 2\pi, \text{ the marking is again } D_{\theta_2} \text{ times the original}
\end{align*}
\]
This completes our description of Hechel-Nielsen coordinates.

Define \( \mathcal{I} = \left\{ \text{isotopy classes of homotopically nontrivial, unoriented, simple closed curves on } M \right\} \).

For \( c \in \mathcal{I} \) define \( \ell_c : \mathcal{Y}(M) \to (0, \infty) \)

\[
(\mathcal{X}, m) \mapsto \inf \left\{ \text{length}(c') \mid c' \in X \text{ homotopic to } m(c) \right\}
\]

**FACT:** \( \ell_c((\mathcal{X}, m)) \) is always realized by the length of a simple closed geodesic \( c' \in X \) homotopic to \( m(c) \).

Let \((0, \infty)^\mathcal{I}\) denote the space of maps \( \mathcal{I} \to (0, \infty) \) with the topology of pointwise convergence (aka the product topology).

The we have

\[
\ell_* : \mathcal{Y}(M) \to (0, \infty)^\mathcal{I}
\]

\[
(\mathcal{X}, m) \mapsto \{ c \mapsto \ell_c((\mathcal{X}, m)) \}
\]

**Thm (Thurston):** \( \ell_* \) is a homeomorphism onto its image.

This homomorphism is proper.

There is true. Let \( \pi : (0, \infty)^\mathcal{I} \to \mathcal{P}(0, \infty)^\mathcal{I} \) denote projection.

**Thm (Thurston):** \( \pi \circ \ell_* \) is a homeomorphism onto its image.
Recall the definition of a measured singular foliation, \( \mathcal{F} \), on \( M \).

For a s.c.c. \( \alpha \in M \) define

\[
\int_{\mathcal{F}} \mathcal{F} = \sup \left\{ \sum_{\alpha_i} \text{measure}(\alpha_i) \mid \alpha_i \text{ disjoint open subarcs of } \alpha \text{ transverse to } \mathcal{F} \right\}
\]

\( = \) (total variation on the measure of \( \mathcal{F} \) restricted to \( \alpha \)).

and for \( c \in \mathcal{F} \) define \( I(\mathcal{F}, c) = \inf_{\alpha \in \mathcal{F}} \int_{\mathcal{F}} \).

I stands for intersection. It's possible for \( I(\mathcal{F}, c) = 0 \), e.g. if \( c \) is a closed leaf of \( \mathcal{F} \).

Recall \( \mathcal{M}_\mathcal{F} \) is the set of measured singular foliations modulo \( \mathcal{S} \) equivalences:

- isotopy
- Whitehead equivalence

Claim: \( I : \mathcal{M}_\mathcal{F} \times \mathcal{S} \to [0, \infty) \) is well-defined.

Taken together, these define a map

\[
I_\ast : \mathcal{M}_\mathcal{F} \longrightarrow [0, \infty)^\mathcal{S}
\]

Thm(Thurston): \( I_\ast \) is injective with image disjoint from \( 0 \).
Use $I_*$ to define a topology on $M^T$. Let $PM^T$ denote projective classes of measured singular $M^T$.

Thm (Thurston): $\pi^* I_*: M^T \rightarrow P[0,\infty)^\#$ induces a map

$$PM^T \rightarrow P[0,\infty)^\#$$

that is a homeomorphism onto its image.

Moreover, $I_*(PM^T) \cong S^{6g-7}$.

Thm (Thurston): Consider $\pi^* l_*(Y)$, $I_*(PM^T) \subset P[0,\infty)^\#$.

1. $\pi^* l_*(Y) = I_*(PM^T)$

2. $(\pi^* l_*(Y)) \cup (I_*(PM^T))$ is homeomorphic to a closed ball.

3. $\text{Mod}(M)$ acts on this closed ball by homeomorphisms.
Recall definitions of \( \mathcal{Y}(M) \), \( \mathcal{A} \), \( \mathcal{I} \), \( \mathbb{I} \), \( \mathbb{I} : [0, \infty)^{\mathcal{A}} \to \mathcal{P}(\mathbb{I}) \).

\( \text{Mod}(M) \) acts on \( \mathcal{A} \) in the obvious way: \( \varphi \cdot c \coloneqq \varphi(c) \).

Then \( \text{Mod}(M) \cap [0, \infty)^{\mathcal{A}} \) acts as: \( (c \varphi \cdot f)(c) = f(c \varphi^{-1} \cdot c) \).

With this, \( \mathcal{I} \) and \( \mathcal{I} \) are equivariant. Let's check \( \mathcal{I} \).

\[
\mathcal{I}(\varphi(X,m)) = \mathcal{I}(X, m \varphi^{-1}) = \left\{ c \mapsto \mathcal{I}(X, m \varphi^{-1}) \right\}
\]

\[
= \left\{ c \mapsto \mathcal{I}(X, m \varphi^{-1}) \right\} = \varphi \cdot \mathcal{I}(X,m).
\]

Recall \( \mathcal{I} \) is an embedding. This means a surface is determined by the lengths of its \( \mathcal{I} \). \( \pi \circ \mathcal{I} \) is also an embedding, meaning it's impossible in hyperbolic geometry to simultaneously expand all the \( \mathcal{I} \) by the same factor.

To better understand \( \mathcal{I}_\ast \), let's begin with intersections of \( \mathcal{I} \).

\[
i : \mathcal{A} \times \mathcal{A} \to [0, \infty)
\]

\[
(b, c) \mapsto i(b, c)
\]

is the min \( \# |b' \cap c'| \). FACT: If \( b' \cap c' \) are closed geodesics in some hypermetric on \( M \) then \( i(b, c) = \# |b' \cap c'| \).

With this, we can define \( \mathcal{I}_\ast : \mathcal{A} \to [0, \infty)^{\mathcal{A}} \) and \( \mathcal{I}_\ast : \pi \circ \mathcal{I}_\ast \).

\( \mathcal{I}_\ast(\mathcal{A}) \subset \mathcal{P}(\mathbb{I}) \) has a slightly surprising topology.
Claim: $i_\ast(D^n_a(b)) \rightarrow i_\ast(a)$.

Pf: Consider $\frac{1}{n} i_\ast(D^n_a(b)) \leq [0, \infty)^d$.

Pick $c \notin \mathcal{A}$. Suppose $i(a, c) = 0$.
Then $i(\frac{1}{n} D^n_a(b), c) = \frac{1}{n} i(b, D^{-n}_a(c)) = \frac{1}{n} i(b, c) \rightarrow 0$.

Next assume $i(a, c) = k > 0$. Examine a small annular nbhd. of $a$.

\[ \frac{1}{n} i(D^n_a(b), c) \leq \frac{1}{n} i(b, c) + \frac{1}{n} \cdot i(na, c) \rightarrow k. \]

We'll next describe an embedding $\mathcal{A} \rightarrow M^1 \mathcal{F}$. Pick $c \notin \mathcal{A}$.
Choose a mini'l graph $G \subset (M - c)$ such that $(M - c)$ is homeo.
to a small neigh. of $G$ in $M - c$. (Minimality ensures there
are no spurious edges, e.g. \[. \]
Then $M - G$ is an annulus. Choose a homeomorphism
$(M - G) \rightarrow [0,1] \times S^1$ taking $c$ to $\frac{1}{12} \frac{1}{3} \times S^1$. Use
the homeo to pull back the "vertical" foliation of the
annulus to $M - G$, so $c$ is a closed leaf. Pull back
the transverse measure of $[0,1]$. Add the singular leaves
$G$ to obtain a measured singular foliation $\mathcal{F}_c$.
The choice of $G$ was not canonical.
Prop: All choices of $G$ result in Whitehead equivalent measured singular foliations.

(Pf in F-L-P 5.3.)

We therefore have a set map $\mathcal{F} \to \mathcal{M}^X$. It's not too hard to believe this map is injective. In fact, we have the following commutative diagram of injections:

$$
\begin{aligned}
\mathcal{F} & \xrightarrow{i_x} [0,\infty)^\mathcal{F} \\
& \downarrow \circ \\
& \mathcal{M}^X \\
& \mathcal{M}^X
\end{aligned}
$$

FACT: $i_x(\mathcal{F}) \subseteq \cap_{x \in \mathcal{F}} (\mathcal{M}^X)$ is dense. Up to scaling, any meas. singular foliation is near a simple closed curve.

Here are some pictures explaining how to get $\tilde{M}_c$ from $c$.
Now we're left with an annulus, which we foliate.

Glue up to obtain $\hat{Y}_C$
Claim: \( \mathcal{F} \hookrightarrow \mathcal{P}(\mathbb{R}^d) \) has dense image, i.e. for measured singular foliation \( \mathcal{F} \), \( \exists \) sequence \( \{ \alpha_n \} \) s.t. for any test curve \( b \) in \( \mathcal{F} \), we have
\[
\alpha_n \cdot \iota(b, c_n) \rightarrow I(\mathcal{F}, b)
\]
for some sequence of scaling factors \( \alpha_n \).

I won't prove this, at least not now.

How can a sequence of hyperbolic surfaces in \( \mathbb{H} \) converge to a simple closed curve in \( \mathcal{P}(\mathbb{R}^d) \)? Let's give a simple example. Recall the Fenchel-Nielsen coordinates on \( \mathbb{H} \). We have \( (\alpha, \beta, \gamma, \delta) \) as a basepoint \( (X(a, b, c, \delta), \text{id}) \in \mathbb{H} \) as a basepoint.

Consider the sequence \( X_n := X(\alpha_n, b_n, c, \delta) \). (Since we're not twisting, let's ignore markings, which will all be \text{id}.)
A little hyperbolic geometry shows that for \( n \to \infty \), \( X_n \) looks like

\[
\psi \rightarrow \text{long thin annulus of height } R \to \infty
\]

the geometry of this piece stays "bounded"
Let's call the curve of length $b$ $\gamma$. (Sorry for the bad notation.)

If $\alpha \in \mathcal{A}$ satisfies $i(\alpha, \beta) = 0$ then $\alpha$ will stay out of the annulus and $l_\alpha(X)$ remains bounded.

If $i(\alpha, \beta) = k > 0$ then $k \cdot l_\alpha(X) + \text{constant} \leq l_\alpha(X_n)$.

So consider the sequence $\frac{1}{R} l_\alpha(X_n) \leq [0, \infty)$.

$$\frac{1}{R} l_\alpha(X_n) \rightarrow i(\alpha, \beta) = i_\star(\beta).$$

$\Rightarrow X_n \rightarrow \beta$ in $P([0, \infty))$.

This is the simplest example of a seq of hyper. surfaces converging to a scc.

The next goal is to show how this works for the humble flat torus $T^2$. Since a torus can be scaled to produce new flat tori, we need a slightly different def'n of $Y(T^2)$.

$$Y(T^2) = \{ (X, m) \mid X \text{ flat 2-torus with a homeo} \}_{m : T^2 \rightarrow X \text{ s.t. } l_{e_1}(X) = 1}$$

It is convenient to orient $e_i$ of $T^2$, but as elements of $\mathcal{A}$ they are unoriented.

$$(x_1, y_1, \ldots, y_m) \in \mathbb{R}^m \times R^2 \mid \text{v}_1 = (1, 0)$$

$$\Rightarrow (x_1, y_2) \in R^2 \mid y > 0$$

$$\uparrow \text{better}$$
The bilipschitz metric I defined on \( \mathbb{R}(T^2) \) works here, \( \bowtie \) reproduces the usual Euclidean topology.

What is \( \bowtie \)?

\[
\bowtie = \begin{cases} 
\pm (m,n) \in \mathbb{Z} \times \mathbb{Z} /\text{sign} & \text{If } m \neq 0 \neq n \text{ then } \\
& \gcd(m,n) = 1, \\
& \text{otherwise } (m,n) \not\in \{(0,1),(1,0)\}. 
\end{cases}
\]

On \( T^2 \) define a 3rd scc \( e_3 \) as shown.

A curve \( c \in \bowtie \) can be written

\[ c = \pm (m,n) = \pm (me_1 + ne_2) \]

(To make sense of this we think of \( e_2 \in \mathbb{R}^3 \)).

Exercise: \( i(c,e_1) = \pm m \), \( i(c,e_2) = \pm n \), \( i(c,e_3) = \pm m - n \).

Moreover, if \( b = \pm (m'e_1 + n'e_2) \) then

\[ i(b,c) = \left| \det \begin{pmatrix} m & n \\ m' & n' \end{pmatrix} \right|. \]

Notice any pair of numbers from \( \{i(c,e_j)\}_{j=1}^3 \) specifies two elements of \( \bowtie \), namely \( \pm (m,n) \) and \( \pm (m',-n) \). To obtain uniqueness all three numbers are required.

Consider the triangular regular simplex of \( \mathbb{R}^3 \):

In barycentric coords \( \frac{1}{3}(x,y,z) \mid x+y+z=1 \), the region where all 3 triangle inequalities \( \begin{cases} x+y \geq z \\
+ y \geq x \\
y + z \geq x \end{cases} \) hold is shown on the right in red. Call it \( \nabla \).
Let \( \text{Cone}(\nabla) := \{ r \cdot (x, y, z) \mid (x, y, z) \in \nabla + r \geq 0 \} \)
and \( \text{Cone}(\partial \nabla) := \{ r \cdot (x, y, z) \mid (x, y, z) \in \partial \nabla + r > 0 \} \).

Notice \( \text{Cone}(\partial \nabla) \) is the set of points where at least one triangle inequality is an equation.

Define
\[
i_3 : \mathbb{A} \longrightarrow \text{Cone}(\partial \nabla).
\]
\[
c \longmapsto (i(e_1, c), i(e_2, c), i(e_3, c)).
\]

\( i_3 \) is injective. Also \( \mathbb{Q} \cdot \text{image}(i_3) = \mathbb{Q}^3 \cap \text{Cone}(\partial \nabla) \).

So \( \mathbb{Q} \) forms the rational points of \( \text{Cone}(\partial \nabla) \).

Similarly \( l_3 : \mathcal{M} \longrightarrow \text{Cone}(\text{int}(\nabla)) \)
\[
X \longmapsto (l_{e_1}(X), l_{e_2}(X), l_{e_3}(X))
\]

This is always 1.

By trigonometry, \( l_3 \) is injective.

Let \( \pi : [0, \infty)^3 \longrightarrow P([0, \infty)^3) \cong \text{simplex} \)
be projection.

(Here we're cheating slightly by
silently ignoring the origin)

Then \( \pi \circ l_3 : \mathcal{M} \longrightarrow P([0, \infty)^3) \) is a homeomorphism of the interior of \( \nabla \).

(This is again trigonometry.)
Lemma: For $c \in \mathbb{R}$, there exists a map $\varphi_c : \text{cone}(\partial \mathcal{A}) \to [0, \infty)$ such that:

- $\varphi_c$ is continuous
- $\varphi_c(\lambda \hat{x}) = \lambda \varphi_c(\hat{x})$ for $\lambda > 0$
- $\iota(b, c) = \varphi_c(\iota(b, e_1), \iota(b, e_2), \iota(b, e_3))$

Proof sketch: Let $c = \pm (m, n) = \pm (me_1 + ne_2)$. Then, let $(x_1, y_1, z) \in \text{cone}(\partial \mathcal{A})$ define

$$
\varphi_c(x_1, y_1, z) := \begin{cases}
\det \begin{pmatrix} y & -x \\ m & n \end{pmatrix} & \text{if } x_3 = x_1 + x_2 \text{ (top edge)} \\
\det \begin{pmatrix} y & x \\ m & n \end{pmatrix} & \text{otherwise}
\end{cases}
$$

Let's check a case. If $m, n > 0$ and $b = x_1 e_1 + x_2 e_2$ for $\beta > \alpha > 0$ then the barycentric coords of $b$ are $(\iota(b, e_1), \iota(b, e_2), \iota(b, e_3)) = (\beta, \alpha, \beta - \alpha)$. So $x_3 \neq x_1 + x_2$.

$$
\varphi_c(\beta, \alpha_1, \beta - \alpha) = \det \begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix} = \iota(b, c).
$$

Use $\varphi_c$ to define $\Phi : \text{cone}(\partial \mathcal{A}) \to [0, \infty)^3$

$$
\begin{array}{c}
(x_1, x_2, x_3) \\
\mapsto \\
\varphi_c(x_1, x_2, x_3)
\end{array}
$$

$\Phi$ is injective. Recall $\iota_* : \mathcal{A} \to [0, \infty)^3$.

By construction $\iota_* = \Phi \circ \iota_3$.

$\Phi$ continuous $\Rightarrow$

$$
(\iota \circ \Phi)(\text{cone}(\partial \mathcal{A})) \subset \mathcal{A} \subset \mathcal{P}([0, \infty)^3).
$$
Claim: The closure of $\mathcal{F}$ in $\overline{P([0,\infty)^d)}$, i.e. $\overline{(\pi\circ \Phi)(\text{cone}(\mathcal{A}))}$, equals the image $(\pi\circ \Phi)(\text{cone}(\mathcal{A}))$.

**Pf:** Suppose $\{c_n\} \subset \mathcal{F}$ s.t. $(\pi\circ \Phi)(c_n)$ converges. Then $\exists \lambda_n > 0$ s.t. $\lambda_n i(c_n) \not\in$ converges to (nonzero) $g \in [0,\infty)^d$.

$\Rightarrow \exists b \neq 0$, $\lambda_n i(c_n, b) \rightarrow g(b)$.

We want to show $g \in \text{image}(\Phi)$.

$g(b) = \lim \lambda_n i(c_n, b) = \lim \lambda_n \Phi_b(i(c_n, e_1), i(c_n, e_2), i(c_n, e_3))$

$= \Phi_b(g(e_1), g(e_2), g(e_3)) = \Phi(g(e_1), g(e_2), g(e_3))$.

Must check that $(g(e_1), g(e_2), g(e_3)) \in \text{cone}(\mathcal{A})$, but OK. \(\Box\)

This shows $\mathcal{F} \cong S^1$ in $\overline{P([0,\infty)^d)}$.

What about singular measured foliations on $T^2$? Let's take a minute to back up and study them.
Thm (Euler-Poincaré Formula): Let $M$ be a closed surface with a measured singular foliation $\mathcal{F}$ and singular set $S = \{p_1, p_2, \ldots, p_n\}$. For each $p_i$ let $k_i \in \{3, 4, 5, \ldots\}$ be the number of leaves coming out of $p_i$. (For example, in the picture $k_i = 4$.) Then

$$2 \chi(M) = \sum_{i=1}^n (2 - k_i).$$

Proof: (See F-L-P, Ch 5.1.6.)

Cor: If $M$ is a torus then $S = \emptyset$.

Let's examine the torus case. Let $M = T^2$. Fix $\mathcal{F}$ on $T^2$.

**FACT:** $\exists$ homotopically nontrivial s.c.c. $c : T^2 \to T^2$ transverse to $\mathcal{F}$.

"Pf:" Look for a recurrent orbit in the transverse direction. $\square$

Cut $T^2$ along $c$ to get an annulus $\mathcal{F}$ is everywhere transverse to $c \Rightarrow$ a leaf cannot begin and end on the same side of the annulus.

So (with some work) one can show the annulus is homeomorphic to $[0,1] \times S^1$ with leaves $[0,1] \times \mathcal{F}^1$. A gluing homeomorphism $g : S^1 \to S^1$ recovers $T^2$ $\mathcal{F}$.

After possibly reparametrizing.
the $S^1$ factor we can assume $g$ is rotation by some angle $\theta_0$. After gluing by $g$ we obtain coords $S^1 \times S^1$ for $T^2$ where $c = \mathcal{I}^0 \times S^1$. In these coords, the universal cover looks like:

![Diagram]

The leaves are straight lines with slope $\frac{\theta_0}{2\pi}$.

Suppose $T^2$ can equipped with a marking and universal cover $\mathcal{I}$ linear map

conjugating one universal cover to the other taking $c$ to $e_3$.

So $\mathcal{I}$ will be taken to straight lines in the $\mathbb{R}^2$ coordinate system also.

Conclusion: $\mathcal{I}$ is isotopic to a foliation on $T^2 = \mathbb{R}^2 / \langle (1, 0, 1, 0) \rangle$ with straight leaves.

So as a set, and ignoring measure, the foliations on $T^2$ up to isotopy is just $\mathbb{RP}^1 = \mathbb{Z}_{\text{slopes}} \circ \mathcal{I}$. The measure of $\mathcal{I}$ is just determined by the induced measure on the transverse curve $c$. After possibly another coordinate change we may assume this measure is $|d\Theta|$, so it's only invariant is the total mass. Thus, as a set

$\mathcal{M}^\mathcal{I} = (0, \infty) \times \mathbb{RP}^1$. 


How did this picture fit into our previous framework (e.g. \( i_3, l_3, \Phi \))? Define

\[
\begin{align*}
I_3 : M^{\mathcal{F}}(T^2) & \longrightarrow \text{Cone}(\partial V) \\
\mathcal{F} & \longmapsto (I(\mathcal{F}, e_1), I(\mathcal{F}, e_2), I(\mathcal{F}, e_3))
\end{align*}
\]

where \( \mathcal{F} \) is as shown:

It's easier to compute \( I(\mathcal{F}, e_i) \) if we tilt this picture, making \( \mathcal{F} \) horizontal:

Then

\[
\begin{align*}
I(\mathcal{F}, e_1) &= |\sin \theta| \\
I(\mathcal{F}, e_2) &= |\cos \theta| \\
I(\mathcal{F}, e_3) &= |\cos \theta - \sin \theta|.
\end{align*}
\]

So \( I_3 \) has image in \( \text{Cone}(\partial V) \), as claimed.

Recall the topology on \( M^{\mathcal{F}} \) was defined by assuming the \( I(\mathcal{F}, \cdot) \) facts are continuous. So \( I_3 \) is, in fact, a homeomorphism.
Recall $\Phi : \text{cone}(\mathcal{A}) \to [0, \infty)^{\mathbb{R}}$, and our thickening construction $\mathcal{A} \to \mathcal{Y}_c \rightarrow \text{cone}(\mathcal{A})$.

How do $I_3 \circ t$ and $i_3$ relate? Fix $c = (m, n) \in \mathcal{A}$ and $t(c) = \mathcal{Y}_c$. Then the angle of $\mathcal{Y}_c$ is $\Theta = \arctan \left( \frac{n}{m} \right)$.

So $I_3(\mathcal{Y}_c) = (|\sin \Theta|, |\cos \Theta|, |\cos \Theta - \sin \Theta|)$

$$= \frac{1}{\sqrt{m^2 + n^2}} (|m|, |m|, |m-n|)$$

and we see $(I_3 \circ t)(c) = \sqrt{m^2 + n^2} \cdot i_3(c)$. They're projectively equivalent. This gives the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{i_3} & \text{cone}(\mathcal{A}) & \xrightarrow{\Phi} & [0, \infty)^{\mathbb{R}} \\
\downarrow t & & \downarrow & & \downarrow \pi \\
\mathcal{Y}_c & \xrightarrow{I_3} & \text{cone}(\mathcal{A}) & \xrightarrow{\Phi} & [0, \infty)^{\mathbb{R}} \\
\end{array}
$$
Introduction to the curve complex of the torus.

Let $\mathcal{D}$ be the set of isotopy classes of non-oriented simple closed curves on the torus $T^2$. Recall

$$\mathcal{D} = \left\{ \pm (0,1), \pm (1,0) \right\} \cup \left\{ \pm (m,n) \mid \gcd(m,n) = 1 \right\} \subset \mathbb{Z} \times \mathbb{Z} \backslash \{0\}$$

and

$$i(\pm (m,n), \pm (m',n')) = \left| \det \begin{pmatrix} m & m' \\ n & n' \end{pmatrix} \right|$$

Imagine $\mathcal{D} = \partial \mathbb{H}^2 = \partial \mathbb{H}^2 = \partial U \cup \partial \infty$. The boundary of the upper half-plane.

Turn $\mathcal{D}$ into a graph. Add an edge between

$$(m,n) \sim (m',n') \iff i((m,n), (m',n')) = 1,$$

i.e. if

$$\exists \text{ a homeo of } T^2 \text{ taking our pair to } e_1 = (1,0) \text{ or } e_2 = (0,1).$$

Note that any two classes in $\mathcal{D}$ have positive intersection, so intersecting exactly once is minimal.

Call the resulting graph $C(T^2)$, the graph of curves on $T^2$. $C(T^2)$ is also known as the Farey graph.

Embed $C = C(T^2)$ into the upper half-plane $\mathbb{H}^2$ by making the edges bi-infinite geodesics between points in $\mathcal{D} = \mathbb{R} \subset \mathbb{H}$.

The curve $\pm (1,0) = \infty$ is joined to $\left( \begin{pmatrix} 1 & m \\ 0 & n \end{pmatrix} \right) = \pm 1$

$$\Rightarrow \pm (m,n) = \pm (m,1) = m \in \mathbb{Z}.$$ Similarly, $\pm (0,1) = 0$ is joined to $\pm (m,1) = \pm \frac{1}{m}$. 


A finite piece of $C \subset H^2$ looks like:

By defn, $C$ is $\text{PSL}_2 \mathbb{Z} = \text{Mod}(\Gamma^2)$-invariant.

Note $\text{PSL}_2 \mathbb{Z}$ acts transitively on $\mathbb{P}^1$, implying $C$ is homogeneous. (Well, at least all the vertices look the same.)

The stabilizer of $\infty = \pm (1, 0)$ equals $(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix})$, which are the Dehn twists along $\infty = \pm (1, 0) = e_1$.

Similarly, the stabilizer of $\frac{p}{q} = \pm (p, q)$ are the Dehn twists about $\pm (p, q)$.

Note $C$ is locally infinite.

Claim: $C$ is connected.

Pf: Let $\gamma$ be the hyperbolic geod. from $t(m_1, n_1) = \frac{m_1}{n_1} \in \mathbb{P}^1$ to $t(m_2, n_2) = \frac{m_2}{n_2} \in \mathbb{P}^1$. Using an elt. of $\text{PSL}_2 \mathbb{Z}$ we may assume $\frac{m_2}{n_2} = \infty$. Then $\gamma$ is a vertical line. $C$ cuts $H^2$ into ideal triangles. Combinatorially $\gamma$ hits these triangles as shown:
Once you see that only a finite number of triangles can hit \( \gamma \), then connectivity follows by simply pushing \( \gamma \) onto the 1-skeleton of the triangles.

Claim: \( C \) has infinite diameter.

pf: Each of the edges of \( C \) crossed by \( \gamma \) separate \( C \). So any path from \( \frac{m_1}{n_1} \) to \( \frac{m_2}{n_2} \) must traverse the edges marked with ticks, possibly at a vertex.

Going from \( \frac{m_1}{n_1} \) to \( \frac{m_2}{n_2} \) in \( \text{Mod}(T^2) \) involves doing Dehn twists about the curves represented by the vertices circled in red, the so-called " pivots." Each pivot adds 1 to the distance. So path with many pivots must be very long.

Notice performing a large number of Dehn twists at a fixed pivot does not increase the distance.
Let's formalize these ideas a little. Let \( V_+ \) and \( V_- \) be distinct points in \( \mathbb{R} \cup \mathbb{S} \).

If \( V_+ \in \mathbb{S} \cup \mathbb{R} \), then think of it as an elt. of \( \mathbb{S} \).

In \( \mathbb{S} \), \( V_+ \) and \( V_- \) by a hyperbolic geodesic.

Let \( E(V_+, V_-) \) denote the set of edges of \( C \) separating \( V_+ \) and \( V_- \). Define an order on \( E(V_+, V_-) \):

\[ e < f \iff e \text{ separates } V_- \text{ from the interior of } f. \]

A pivot is a vertex of \( C \) shared by \( \geq 1 \) pair of consecutive edges of \( E(V_+, V_-) \).

(The pivots are circled in red in the picture on the previous page.)

A block of pivot \( p \) is a max'l set of consecutive edges \( e_1 < e_2 < \ldots < e_{w(p)} \) all sharing the vertex \( p \). \( w(p) \) is the width of the block.

\[ \text{Assume for now that } V_+, V_- \in \mathbb{S}. \]

A shortest path from \( V_- \) to \( V_+ \) must intersect each edge of a block, possibly through a vertex. If \( w(p) > 3 \) then a shortest path will intersect the block in \( e_1 \) and/or
In all cases a shortest path will first intersect the block, stay within a (closed) $\frac{1}{2}$-neighborhood of the block, and leave the block.

Prop. (Minsky?): $C$ is Gromov hyperbolic.

Recall the def'n.

Def: Let $X$ be a geodesic metric space. $X$ is Gromov hyperbolic if there exists $\delta > 0$ with the following property: join $x, y, z \in X$ by shortest paths. Then the path from $x$ to $z$ is contained in the union of the $\delta$-neighborhoods of the paths from $x$ to $y$ and $y$ to $z$.

Pf of Proposition: It suffices to consider a triple $x, y, z \in C$ and shortest paths $[xy]$, $[yz]$, and $[xz]$ in $C$. Let $e$ be an edge of $E(x, z)$.

Either: (i) $e \in E(x, y)$ or (ii) $e \in E(y, z)$ or (iii) $y$ is a vertex of $e$.

In case (i), $[xy]$ must hit $e$.
In case (ii), $[yz]$ must hit $e$.
In case (iii), $y$ is in $e$. 

If $w(p) \leq 3$ then a shortest path may avoid $p$. 

So $e$ is inside a (closed) $1$-neigh. of $[xy] \cup [yz]$. We now use the fact that $[xz]$ is inside a (closed) $\frac{3}{2}$-neigh. of $[xy] \cup [yz]$. ⊠

Being Gromov hyperbolic is good for many reasons. Consider a pair of infinite hyperbolic geodesics beginning at $0$ and terminating at $y_1, y_2 \in \mathbb{R} - \mathbb{Q}$. These geods. determine ordered edge sequences $[e_1, \ldots, e_n]$ and $[e_1', \ldots, e'_n]$. As soon as the sequences $e_1, e_2, e_3, \ldots$ and $e_1', e_2', e_3', \ldots$ see an unequal pair, then the corresponding geodesics in $C$ will begin to diverge linearly. (From the model on page 4 for shortest paths we see shortest paths are unique except for finite ambiguity in traversing blocks of width $\leq 3$.) From this it follows that the Gromov boundary of $C$ is $\mathbb{R} - \mathbb{Q}$. Notice that $\partial C \cup C$ looks like $\mathbb{R} \cup \mathbb{Q}$, but the topology is very different. In particular, it is not compact.

For example, the sequence $\{e_n\}_{n \in \mathbb{N}} \in C$ has no convergent subseq. In a sense, this is the only way a sequence can diverge: by having lots of twisting. An element of $\partial C = \mathbb{R} - \mathbb{Q}$ corresponds to a measured foliation of irrational slope.
Discuss axes of hyperbolic elements of $\text{PSL}_2 \mathbb{Z} = \text{Mod}(T^2)$.

What's the connection between $C(T^2)$ and $Y(T^2)$?

Recall $Y(T^2) = \left\{ (x,y) \mid y > 0 \right\}$.

For $c \in \mathbb{R}$ let

$$U_c = \left\{ (x,y) \in Y(T^2) \mid \frac{\text{length}(c)}{\text{area}} \leq \varepsilon \right\}.$$

This is a torus where $c$ is not long (it will not be very small, just a medium size).

For example

$$U_\infty = \left\{ (x,y) \mid \frac{\text{length}(c)}{\text{area}} = \frac{\text{length}((1,0))}{\text{area}} = \frac{1}{y} \leq \varepsilon \right\}$$

$$= \left\{ (x,y) \mid y \geq \frac{1}{\varepsilon} \right\}.$$

Apply $\text{Mod}(T^2) = \text{PSL}_2 \mathbb{Z}$ to $U_\infty$ and conclude that $U_c$ for $c \neq \infty$ is a horodisk tangent to $c$.

FACT: $\sup \sum_{c \in \mathbb{R}} |U_c|$ are pairwise disjoint \( \sum_{c \in \mathbb{R}} |U_c| = 1 \).

If $\varepsilon = 1$ then it looks something like

$$U_\infty$$

$$0 \, 1 \, 2 \, \mathbb{R}$$
Consider the nerve of the collection of \( n \) sets \( \{ U_c \}_{c \in \mathbb{R}} \). This nerve is a graph with vertices \( U_b \cup U_c \) joined by an edge iff \( U_b \cap U_c = \emptyset \). (The higher skeleta are empty.) This nerve is exactly \( C(T^2) \). This is not an accident. The collection \( \{ U_c \}_{c \in \mathbb{R}} \) is often called the thin parts of Teichmüller space. Then \( C \) is the nerve of the thin parts. This will persist in higher genera.
Subsurface projections in the case of the torus

Let $\mathcal{S}$, as usual, be the set of isotopy classes of (embedded) homotopically nontrivial simple closed curves. Recall $\mathcal{S}$ is naturally identified with $PQ = \mathcal{Q}_0 \cup \mathcal{Q}_2$. For $c \in \mathcal{S}$, $T^2 - c$ is an annulus. We need the def'n of the curve complex of an annulus, which is annoyingly complex. For the sake of culture I'll give the official def'n from Masur–Minsky's "Geometry of the complex of curves. II." For annulus $A$, let the vertices $C(A)$ be the set

\[
\left\{ \text{paths from one boundary component of } A \text{ to the other} \right\} \cup \left\{ \text{homotopies fixing the boundary pointwise} \right\}.
\]

E.g. These are the same vertex:

These are not:

Obviously this set $C(A)$ is huge, but we're stuck with it.
Join the vertices of \( C(A) \) by an edge if they have representatives with disjoint interiors, forming the curve complex of the annulus.

For \( \alpha, \beta \in C(A) \) vertices, the signed algebraic intersection number \( \alpha \cdot \beta \) is well defined. (Only count interior intersections.)

**Lemma:** \( d_{C(A)}(\alpha, \beta) = 1 + |\alpha \cdot \beta| \)

**Lemma:** \( \gamma \cdot \alpha = \gamma \cdot \beta + \beta \cdot \alpha + \epsilon \) for \( \epsilon \in \mathbb{Z} - \{1, 0, -1\} \).

(Note \( \gamma \cdot \alpha = -(|\alpha \cdot \gamma|)! \))

Picking a base \( \alpha \in C(A) \) defines a map \( \tau: C(A) \to \mathbb{Z} \)

\[
\beta \mapsto \beta \cdot \alpha.
\]

**Lemma:** \( \tau \) is a quasi-isometry, namely

\[
|\tau(\beta) - \tau(\gamma)| \leq d_{C(A)}(\beta, \gamma) \leq |\tau(\beta) - \tau(\gamma)| + 2.
\]

So without losing anything one can imagine \( C(A) \) is simply \( \mathbb{Z} \) (as a metric space, not a group).

(Curves on \( T^k \) compact)

**Given** \( c \in \mathbb{S} \) let \( A_c \), be the annulus obtained in the obvious way from \( T_c \). For \( b \in \mathbb{S} - \{c\} \) define the projection \( \pi_c(b) \) to \( C(A_c) \) as the set:

\[
\{ \beta \mid \beta \text{ equals the restriction to } A_c \text{ of } \exists b' \text{ for some s.c.c. } b' \text{ isotopic to } b \} \subset C(A_c).
\]

Then \( \pi_c(b) \) is a set of diameter 1.
To make this more explicit, let $c = \pm (1,0) = \frac{1}{\delta} = \infty \in \mathbb{F}$ and $\alpha = \pm (0,1) = 0 \in \mathbb{F}$.

Then $b = \pm (p,q) = \frac{p}{q} \in \mathbb{F}$ will project to the annulus $A_c$ as $q$ curves $b_i \mathbb{Z}$ of slope $\frac{p}{q}$. (I guess we should have taken reciprocals somewhere.) Each $b_i$ will intersect $\alpha$ either $\left\lfloor \frac{p}{q} \right\rfloor$ or $\left\lceil \frac{p}{q} \right\rceil$ times. So the projection $\pi_c$ can safely be thought of as taking the integer part of $\frac{p}{q}$.

**Prop (Bounded Geodesic Image):** Suppose $g$ is a geodesic in $\mathbb{C}(T^2)$ disjoint from $c \in \mathbb{F}$. Then the diameter of the projection $\pi_c(g)$ is at most 4.

**Pf:** Wolog assume $c = \frac{1}{\delta} = \infty$. Assume $\exists b_1, b_2 \in \mathbb{F}$ s.t. the diameter of $\pi_c(b_1 \cup b_2)$ is $\geq 5$. Then $b_1 = \frac{p_1}{q_1}$ and $b_2 = \frac{p_2}{q_2}$ are separated by at least 4 integer points.

Suppose wolog $b_1 < b_2$ and $k_1$ is the least integer $\geq b_1$, $k_2$ is the greatest $\leq b_2$. Then $g$ must pass through $k_1$ and $k_2$.

\[ b_1 \quad k_1 \quad k_1 + 1 \quad k_1 + 2 \quad k_1 + 3 = b_2. \]
The unique geodesic from $k_1$ to $k_2$ is $\exists k_{2,00}$, $k_{2,0}^2$. By assumption $g$ is disjoint from $\infty$, yielding a contradiction. 

This proposition is true in higher genus, proved by Masur–Minsky.

**The generic curve complex**

We will need to *assume* allow surfaces with boundary. Let $M$ be a compact oriented surface, not $\emptyset$, $\bigcirc$, $\bigstar$, $\bigcirc_{\text{hole}}$, $\bigcirc_{\text{torus}}$, $\bigcirc_{\text{torus with hole}}$.

$\mathcal{C} = \{ \text{isotopy classes of simple closed curves} \}$

Then $C(M)$ is a simplicial complex with a $(k-1)$-simplex given by a pairwise disjoint $k$-tuple of $\mathcal{C}$. Make it a metric space by making each simplex a standard Euclidean simplex $\ast$ using a path metric. We will only consider the 1-skeleton, the curve graph.

**Claim:** The curve graph is locally infinite.

**Pf:** Find $a,b,c \in \mathcal{C}$ s.t. $a \cap b = a \cap c = \emptyset \ast \cup c \neq \emptyset$.

(This is possible because we ruled out the "low genus" cases, spheres with $< 5$ punctures.)
Then the curves $D^0_b(z)$ are all distance 1 from $a \in \mathbb{D}$.

Claim: The curve graph is connected. In fact
\[ d(\alpha, \beta) \leq 2 \cdot i(\alpha, \beta) + 1. \]

Pf: Assume $\#(\alpha \cap \beta) = i(\alpha, \beta)$, i.e. they intersect minimally.
If $i(\alpha, \beta) = 0$ then we're done.
If $i(\alpha, \beta) = 1$ then consider a small neighborhood $U$ of $\alpha \cap \beta$. $U$ is necessarily a punctured torus.
Consider the curve $\partial U$. If $\partial U$ is isotopic into $\partial M$ the $M$ is a punctured torus. We assumed $M$ is not a punctured torus. $\Rightarrow \forall U \in \mathcal{C}(M), d(\partial U, \alpha) = d(\partial U, \beta) = 1$

$\Rightarrow d(\alpha, \beta) = 2.$

Now assume $i(\alpha, \beta) = k \geq 2$ and argue by induction.

Consider a pair of adjacent points in $\alpha \cap \beta$.

\[ \text{in a neighborhood of } \alpha \]

\[ a \text{ neighborhood of } \alpha \]

\[ \partial \text{ neighborhood of } \alpha \]
Case I: Assume $\beta$ can be oriented as in the picture:

Then do a surgery as shown to produce $\beta'$.
Then $\beta'$ must cross $\beta$ exactly once, from the left side to the right, so $i(\beta, \beta') = 1 \Rightarrow d(\beta, \beta') = 2$ and $\beta'$ is not isotopic into $\partial M$.

By induction $d(\alpha, \beta) \leq d(\alpha, \beta') + d(\beta', \beta)$

$\leq 2(k-1) + 1 + 2 = 2k + 1$.

Case II: Assume $\beta$ can be oriented as shown:

Then perform surgery to produce $\beta_1$, $\bar{\beta_2}$.
Each $\beta_j$ is homot. non-trivial, $\rho$

$i(\beta_j, \alpha) \leq k - 2$.

However, $i(\beta, \beta_j) = 0$ so we must show at least one of the $\beta_j$ is not isotopic into $\partial M$.

If $\beta_j$ is not isotopic into $\partial M$ then we’re done by induction. Suppose $\beta_1$ and $\bar{\beta_2}$ are homotopic to components of $\partial M$. Then the component of $M - \beta$ containing the $\beta_j$ must be a thrice-punctured sphere.
Apply the above argument to the other segment of \( \beta \) cut along \( \alpha \) and \( \beta \) containing the point \( p \) as shown. If we end up again in this case then \( M \) must be a 4-times punctured sphere, which we assumed is not the case.

Notice the is no reverse inequality; one cannot bound distance from below by intersection number. E.g.

\[
i(D_c^n(b), b) \xrightarrow{n \to \infty} \infty, \text{ but } d(D_c^n(b), b) \leq d(D_c^n(b), a) + d(a, b) = 2.
\]

So it's not obvious that the curve graph has \( \infty \) diameter.
It's worth noting the geometric meaning of small distances in the curve graph:

\[ d(b, c) = 1 \iff b \cap c \text{ are disjoint} \]

\[ d(b, c) = 2 \iff \exists \text{ a disjoint from } b \cup c \]

\[ d(b, c) > 2 \iff M - (b \cup c) \text{ is a union of disks and annuli. (If } \partial M = \emptyset \text{ then } M - (b \cup c) \text{ is only disks. The annuli always have one boundary component lying in } \partial M \text{.)} \]

Seeing the difference between distance 7 and 70 is not so easy.
The next goal is to prove that the curve complex has infinite diameter. More precisely, we'll show the curve graph has infinite diameter. The argument here is due to T. Kobayashi "Heights of simple loops and pseudo-Anosov homeomorphisms" (Prop 2.2). The argument requires several preliminary facts from Teichmüller theory. Let me state them first, then give Kobayashi's argument. We'll go back and fill in the facts afterward, if we have time.

FACTS:

1. The space of projective measured foliations $PMF$ is compact. (In fact it's a sphere, but we won't need this.)

2. The intersection number $i: \mathbb{A} \times \mathbb{A} \to \{0, 1, 2, \ldots \}$ extends to a continuous fct. $i: IMF \times IMF \to [0, \infty)$.

3. Recall the definition of a pseudo-Anosov homeomorphism. We will need the existence of at least one pseudo-Anosov homeo with unstable lamination $\mathcal{L}$. Moreover, we'll need the following fact that we have not seen at all previously:

$$i(\mu, \mathcal{L}) = 0 \Rightarrow \mu = \mathcal{L}.$$
Let $\mathcal{Y}_c$ be the measured foliation determined by $c \in \mathcal{F}$. From the def'n of pseudo-Anosov it follows that $\mathcal{Y}$ has no closed leaves. In particular, $\iota(\mathcal{Y}_c, \mathcal{Y}) > 0$, $\mathcal{Y}_c \neq \mathcal{Y}$.

4. Recall there is an embedding $\mathcal{F} \hookrightarrow \mathcal{PM}^{\mathcal{Y}}$. We'll use that the image is dense.

**Thm:** The curve graph of $M$ has infinite diameter.

**Pf:** Pick $c \in \mathcal{F}$ and let $Z_n \subset \mathcal{PM}^{\mathcal{Y}}$ be the (compact) closure of the set $\{b \in \mathcal{F} \mid d(b, c) \leq n/2\}$.

As above, let $\mathcal{Y}$ be the unstable foliation of some $\mathcal{Y}$-Anosov homeomorphism. By the remarks in Fact 3 above, $Z_0 = \mathcal{F} \neq \mathcal{F}$. Suppose by induction that $Z_k \cap \mathcal{F} = \emptyset$ for some $k < n$. In search of a contradiction assume $\mathcal{Y} \in Z_n$. Then $\exists b_j \in \mathcal{F}$ s.t. $d(b_j, c) = n$ and $b_j \to \mathcal{Y}$ in $\mathcal{PM}^{\mathcal{Y}}$. Also $\exists a_j \in \mathcal{F}$ s.t. $d(a_j, c) = n - 1$ and $d(b_j, c_j) = 1$, implying $i(b_j, c_j) = 0$. Up to subsequence $a_j \to \lambda \in Z_{n - 1}$. By continuity

$$0 = i(a_j, b_j) \to i(\lambda, \mathcal{Y}) \Rightarrow \lambda = \mathcal{Y}.$$  

Contradiction. \[\therefore \quad \exists \ n, \mathcal{Y} \in Z_n.\]

By Fact 4, $\exists c_n \in \mathcal{F}$ s.t. $c_n \in \mathcal{PM}^{\mathcal{Y}} - Z_n$. Then $d(c_n, c_n) > n$. \[\square\]
To state the general bounded geodesic image theorem we must define subsurface projections.

Suppose \( X \subseteq M \) satisfies:

- \( M \) is a compact oriented connected surface with (possibly empty) boundary. \( \not\equiv \) Assume \( M \not\in \{ \emptyset, \mathcal{O}, \mathcal{I}, \mathcal{A}, \mathcal{O} \} \).
  - (For \( M = \mathcal{O} \) join \( \mathcal{O} \)\( \mathcal{O} \) with intersection number 1.)
  - (For \( M = \mathcal{I} \) join \( \mathcal{I} \)\( \mathcal{I} \) with intersection number 2.)
- \( X \) is a connected proper compact subsurface
  s.t. \( X \hookrightarrow M \) induces an injection \( \pi_1 X \hookrightarrow \pi_1 M \),
  - \( X \) is not freely homotopic into \( \partial M \),
  - \( X \not\equiv \{ \emptyset, \mathcal{O}, \mathcal{A} \} \). \( (X \) also is not \( \emptyset, \mathcal{O} \).)
  - For simplicity, let's assume \( X \not\equiv \{ \emptyset \}. \) \( (This \ case \ is \ pesky.) \)

Define projection \( \pi_X : \mathcal{C}(M) \to \{ \text{subsets of } \mathcal{C}(X) \} \) \( (C \) indicates the curve graph. \)

Pick be \( \mathcal{C}(M) \). \( \not\equiv \) Assume we chose \( b \) to intersect \( \partial X \)
minimally in its homotopy class. If \( b \cap X = \emptyset \) then \( \pi_X(b) = \emptyset \).
If \( b \cap X \) then \( \pi_X(b) \) is simply \( \mathcal{S}b \), thought of as a \( \mathcal{O} \) of \( \mathcal{X} \).
Otherwise let \( \mathcal{S}b_1, b_2, \ldots, b_n \) be the components of \( b \cap X \).
For each \( b_i \) let \( \mathcal{X}_i \subseteq \partial X \) be the components of \( \partial X \)
intersecting \( b_i \), and consider a small \( \mathcal{X} \) closed neighborhood
Let $S_i$ be the components of $\partial K$ that are homotopically nontrivial and not homotopic into $\partial X$. $S_i$ must be nonempty. Let $\pi_X(b) = \bigcup S_i$. The diameter of $\pi_X(b)$ is $\leq 2$.

Note that some components of $S_i$ may be homotopic in $X$, as seen in the picture's example.

[Bounded geodesic image thm (Masur-Minsky): Let $g \subset C(M)$ be a (possibly infinite) geodesic such that $\pi_X(v) \neq \emptyset$ for every vertex $v$ of $g$. Then there is a constant $D(M)$ such that the set $\bigcup_{v \in g} \pi_X(v)$ has diameter at most $D$.]

[Gromov hyperbolicity: $C(M)$ is Gromov hyperbolic.]

For now let's assume a Gromov hyperbolicity with constant $S$ and try to get an intuition for the Bounded Geodesic Image Theorem. (This example is based on one from Masur-Minsky II.)
Suppose $M = \bullet$. Let $g$ be a (long) geodesic segment $\ldots, u, v, w, \ldots$. Let $g'$ be another long geodesic segment with endpoints distance 0 or 1 from the endpoints of $g$. Then $g \& g'$ are $(2 \delta + 1)$-fellow travelers.

(Note that $\pi_X(u) \& \pi_X(w)$ are distance 1 in this picture.)

Let $X \subset M$ be the closure of the 4-punctured sphere component of $M - v$. Then $u, w \subset X$.

Claim: If $\pi_X(u) \& \pi_X(w)$ are far apart in $C^0(X)$ then $g'$ must pass through $v$. (Not the case in the picture!)

Suppose $g'$ does not pass through $v$. Form the red path in $C(M)$ from $u$ to $w$ as shown:

1. go forward along $g$ from $w$ distance $2 \delta + 2$
2. go backward along $g$ from $u$ distance $2 \delta + 2$
3. skip over to $g'$ with a path of length $\leq 2 \delta + 1$ (by fellow traveler property)
4. skip over to $g'$ with a path of length $\leq 2 \delta + 1$
5. Close up with a path along $g'$ of length $\leq 8\delta + 8$ (by the triangle inequality).

Every point of the red path is not $v$. So we can project it to $C(X)$.

Lemma: If $d_{C(M)}(b, c) = 1$ then $\text{diam}(\pi_X(bUC)) \leq 2$.

So $\pi_X(\text{red path})$ can be modified slightly to make a path of length at most $2(2\delta + 2 + 2\delta + 2 + 2\delta + 1 + 2\delta + 1 + 8\delta + 8) = 32\delta + 28$

We see that $d_{C(X)}(\pi_X(u), \pi_X(w)) > 32\delta + 28$

implies $g'$ must pass through $v$.

Assume now that $g'$ passes through $v$.

Let $u', w'$ be as in the picture. Using the red paths, and a similar argument, one can bound $d_{C(X)}(\pi_X(u), \pi_X(u'))$ and $d_{C(X)}(\pi_X(v), \pi_X(v'))$. It follows that $[uv]$ and $[u'w']$ are fellow travellers. All this followed from knowing that the endpoints of $g$ and $g'$ are close to each other.
Geodesics in Teichmüller space and the curve complex

The next goal is to understand how to build (quasi-)geodesics in the curve complex. Given that the curve complex is not locally finite, it's not obvious how to find them.

To begin we must back up to Teichmüller space. Recall the def'n of Teichmüller space.

\[ \mathcal{M}(M) = \left\{ (X, \text{homeom. } m : M \to X) \mid X \text{ hyperbolic surface} \right\} \sim \]

where \( (X, m_X) \sim (Y, m_Y) \iff \exists \text{ isometry } i : X \to Y \text{ s.t. } i \circ m_X = m_Y \).

Sadly, for this part of the story this is the wrong definition. We need the complex analytic definition

\[ \mathcal{M}(M) = \left\{ (X, \text{homeom. } m : M \to X) \mid X \text{ Riemann surface} \right\} \sim \]

where \( (X, m_X) \sim (Y, m_Y) \iff \exists \text{ conformal map } i : X \to Y \text{ s.t. } i \circ m_X = m_Y \).

Next we need to know what a quasiconformal map is, at least in the differentiable case.
Spa \( f: X \to Y \) is differentiable at \( p \in X \). Using only structures on \( X \) and \( Y \), we don't know how long vectors are in \( T_pX \) or \( T_{f(p)}Y \). However, ratios \( \frac{||v||}{||v'||} \) are well defined for \( v \) in \( T_pX \) (or \( T_{f(p)}Y \)). So pick \( v \in T_pX - 0 \) and take the circle \( C \) of vectors \( v \) in \( T_pX \) st. \( ||v|| = 1 \).

Define the quasi conformal constant of \( f \) at \( p \) to be \( \sup_{w_1, w_2 \in C} \frac{||df_p(w_1)||}{||df_p(w_2)||} \) and call it \( K(f,p) \). Note \( K(f,p) \ge 1 \), \( \forall \) \( p \in X \).

Let the quasi conformal constant of \( f \) be the smallest \( K \) st. \( K(f,p) \le K \) a.e. (We're ignoring many analytic details here, but the a.e. is important. We cannot reasonably require \( f \) to be differentiable or most everywhere. That would rule out many interesting examples.)

\[
K(f,p) = \frac{L}{\ell}.
\]
The Teichmüller metric on $\mathcal{Y}(M)$ is

$$d_{\mathcal{Y}}((X,m_x),(Y,m_y)) = \frac{1}{2} \log K$$

where

$$K = \inf \left\{ \text{quasi-conformal constant of } f : X \to Y \text{ s.t. } m_Y \circ f^* = m_X \right\}$$

There is a not-so-obvious fact hiding under here. If $K = 1$ then $d_{\mathcal{Y}} = 0$, so in fact $X$ and $Y$ are conformally equivalent. This means having a q.c. constant of 1 a.e. implies conformality.

Everyone gets to define a metric on $\mathcal{Y}(M)$. Why is this one interesting? Because of Teichmüller's thm., which I'll try to explain next.

Fix $(X,m_x), (Y,m_y) \in \mathcal{Y}(M)$.

Part 1: $\exists f : X \to Y$ s.t. $m_Y \circ f^* = m_X$ and the q.c. constant of $f$ equals $d_{\mathcal{Y}}((X,m_x),(Y,m_y))$. In other words, the infimum of $d_{\mathcal{Y}}$ is uniquely realized. $f$ is called the Teichmüller map.

Part 2: We can describe the Teichmüller geodesic from $(X,m_x)$ to $(Y,m_y)$ as $(X_t,m_x)$ for a 1-parameter family of conformal structures $X_t$, $1 \leq t \leq e^{2d_{\mathcal{Y}}(X,Y)}$.

As follows. Let $X_1 = X$.

$\exists$ a pair of transverse measured singular foliations $\mathcal{F}_h$ and $\mathcal{F}_v$ on $X$.

Notice this is the q.c. constant of $f$. 
(think a horizontal foliation + a vertical foliation) such that at a nonsingular pt. of $X$ the conformal structure of $X_t$ is given by stretching by a factor of $T^t$ in the horizontal direction and squishing for a factor of $T^t$ in the vertical direction.

This defines a path $(X_t, m_x)$ in $\mathcal{M}(M)$ s.t.

$$(X_{t_0}, m_x) = (Y, m_Y), \text{ for } t_0 = e^{2\log(Y, Y)}$$

This means the Teichmüller map is affine away from the common singular pts. of $Y_h$ and $Y_v$.

Summary: The Teichmüller metric has unique geodesics with a fairly explicit model for how the surface is changing.

Mention the bi-infinite Teichmüller geodesic associated to a pseudo-Anosov homeomorphism.
Recall the notion of a Margulis constant $\mu$ for hyperbolic surfaces. We need the fact that, on any hyperbolic surface, two simple closed curves of length $< \mu$ never intersect.

Next I'll define the electric Teichmüller space $Y_{el}(M)$. Start with $Y(M)$ equipped with the Teichmüller metric. Now for each $x \in Y(M)$, add a disjoint point $c$ to $Y(M)$. Finally, if the length of $c$ in $(x, y)$ is less than $\mu$, then join $(x, y) + c$ by an edge. This makes any pair of surfaces where $c$ is short distance $< \bar{d}$ in $Y_{el}$. There is an obvious map $C \to Y_{el}$ taking a curve $c$ to the added point $c \in Y_{el}$. This map is clearly

**FACT:** For any set of pairwise disjoint curves $c_i, c_j$ on $M$, $\exists (x, y) \in Y(M)$ where all the curves $c_i$ are short, i.e. length $< \mu$.

**Corollary:** $C \to Y_{el}$ is $2$-Lipschitz.

**FACT:** There exists $D$ st. the image of $C \to Y_{el}$ is $D$-dense. I.e. there is always a curve of medium length, and it can be shortened without moving too far in $Y(M)$.
Next define $\Phi: Y \rightarrow \mathbb{C}$ of subsets of curves on $X$.

**FACT:** $\exists b \text{ s.t. if } d_Y(X_{mx}, Y_{my}) \leq 1$ then $\text{diam}(\Phi(X) \cup \Phi(Y)) \leq b$.

This implies $\Phi$ is Lipschitz. We'd like to electrify $\Phi$. This is no problem, just define $\Phi_\ell(c) = c$ for the added points. The resulting map $\Phi_\ell: Y_\ell \rightarrow \mathbb{C}$ is Lipschitz, and it is a quasi-inverse to the map $\mathbb{C} \rightarrow Y_\ell$.

We conclude that $\mathbb{C} \rightarrow Y_\ell$ is a quasi-isometry.

**Thm (Masur - Minsky):** The map $\Phi: Y \rightarrow \mathbb{C}$ sends Teichmüller geodesics to quasi-geodesics of $\mathbb{C}$ with uniform quasi-geodesic constants.

This is our desired model of geodesics in $\mathbb{C}(\mathcal{M})$ as images under $\Phi$ of Teichmüller geodesics.
In a related vein, let's finish with a brief introduction to tight geodesics.

Define a geodesic in $\mathcal{C}(\Sigma)$ to be a sequence

$\Sigma_0, \Sigma_1, \ldots, \Sigma_n$ of simplices of $\mathcal{C}(\Sigma)$ s.t.

* for any $i, j$ and any curves $c_i \in \Sigma_i$, $c_j \in \Sigma_j$,

$$d(c_i, c_j) = |i - j|.$$ 

Notice there is a lot of "local" ambiguity in a geodesic of $\mathcal{C}(\Sigma)$. For example, let's return to a common example

$$d(a, b) = 2,$$ but if $c$ is any curve in the right half of the surface (3 are shown) then $[a, c, b]$ is a geodesic. To eliminate this ambiguity, Masur-Minsky introduced the notion of a tight geodesic. For any pair $\Sigma_i, \Sigma_{i+2}$ there is a minimal connected $\mathcal{M}$-injective subsurface $R \subset M$ containing all the curves of $\Sigma_i, \Sigma_{i+2}$. As $d(\Sigma_i, \Sigma_{i+2}) = 2$, we know $R \neq M$. 

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Finally, our geodesic is tight if every curve of $\Sigma_{i+1}$ is homotopic into $\partial R$.

For example, in the above picture the only tight geodesic is $\Sigma a, d, b \Sigma$. Notice tightness is local.

Thm (MM): Between any two points of $C$ there is a finite number of tight geodesics.
Laminations, Train tracks, and singular foliations June 10, 2009

Fix $M$ a closed hyperbolic surface. (To keep things simple today we'll assume $M$ has no boundary.)

**Def:** A geodesic lamination $\lambda$ on $M$ is a union of disjoint geodesics of $M$ forming a closed set.
(Note that a geodesic is either closed or bi-infinite.)

Each component of $M - \lambda$ has area $\geq \pi$, because it has geodesic boundary and one can apply Gauss-Bonnet.

$\Rightarrow M - \lambda$ has $\leq \frac{\text{Area}(M)}{\pi} = 2\cdot |\chi(M)|$ components.

With a little more work one can show $\lambda$ has area 0 in $M$. (For this and much more, see Chapter 8 section 5 of Thurston's Notes.)

With a homeomorphism $\Phi: M \to N$, for $N$ a hyperbolic surface, we can push $\lambda$ to $N$ by identifying a geodesic in $M$ with a distinct pair in $\partial M$, i.e. then using the $\tau_1$-equivariant homeomorphism $\partial M \to \partial N$. So the choice of metric on $M$ is just for convenience.
Def: A measured geodesic lamination $\lambda$ on $M$ is a geodesic lamination $\gamma$ together with a measure $\mu$ on the set of compact arcs of $M$ transverse to $\lambda$ satisfying:

1. $\mu(a) < \infty$ for any compact arc $a$ transverse to $\lambda$

2. If $a_t$ is a 1-parameter family of compact arcs transverse to $\lambda$ s.t. $a_t \cap \lambda = \emptyset$ for all $t$ then $\mu(a_0) = \mu(a_\pm) = t$.

3. We assume $\mu$ has full support, i.e. $a_t \not\equiv \emptyset \Rightarrow \mu(a) > 0$.

Note the multicurve with additional "spiraling" geodesics from page 1 cannot be made into a measured geodesic lamination. A measured geod. lamin cannot have an infinite geod. spiraling into a closed geod.

One could, in this situation, find a family $a_t$ of transverse arcs s.t. $a_1 \not\equiv a_0 \Rightarrow \mu(a_t) = 0$. 
The next goal is to define train tracks. It's best to draw lots of pictures. A branch of track is an embedded square in $M$ with its vertical and horizontal foliation. The horizontal foliation forms the leaves of the branch. The vertical foliation form the ties.

A switch is a union of branches glued along boundary ties so there are no dead-end leaves.

A train track $\mathcal{T}$ on $M$ is a collection of branches and switches so there are no dead-end leaves, and every component of $M - \mathcal{T}$ is diffeomorphic to a 1-gon or a bigon (or a 0-gon).

Note that topology in $M - \mathcal{T}$ is allowed. E.g.
Now assume $\mathcal{A}$ is a geodesic lamination with measure $\mu$. Note that inside any fixed branch of $\mathcal{T}$ the measure of a tie is constant. Moreover, at a switch the total measure of the ties on the left equals the total measure of the ties on the right. This motivates the def'n

**Def.** A weighted train track assigns a positive weight to each branch such that at each switch the total weights on each side are equal.

So our construction builds a weighted train track from a measured geodesic lamination. Notice that by choosing $\varepsilon$ smaller we obtain finer approximations of our lamination.

Building a measured singular foliation from a weighted train track is easy, if a bit technical to nail down. Simply the complement program.

First add some singular leaves to the complement of $\mathcal{T}$ and then collapse the rest of the complement of $\mathcal{T}$ onto the singular leaves.
Add singular leaves if then collapse.

Keep in mind the ties are transverse to the resulting foliation.

There is ambiguity when choosing how to add singular leaves. All choices are Whitehead equivalent. When a complementary region has some topology then adding singular leaves is slightly more complex. We'll skip these details here. This gives a singular foliation. What about the measure? For each branch of the train track of weight \( w \) put a uniform Lebesgue measure on the ties of total measure \( w \). This measure transfers in the obvious way to curves transverse to the singular foliation.

This describes a maps:

\[
\{\text{measured laminations}\} \rightarrow \{\text{weighted train tracks}\} \rightarrow \{\text{measured singular foliations}\}.
\]
We'll complete the picture with a map
\[
\left\{ \text{measured singular foliations} \right\} \rightarrow \left\{ \text{measured laminations} \right\}.
\]

Consider a measured singular foliation $\mathcal{F}$ on $M$. Lift $\mathcal{F}$ to a measured singular foliation $\tilde{\mathcal{F}}$ on $\tilde{M}$. The boundary at infinity $\partial_\infty \tilde{M}$ is $S^1$.

Each smooth leaf of $\tilde{\mathcal{F}}$ lifts to a curve in $\tilde{M}$ with endpoints in $\partial_\infty \tilde{M}$.

**Claim:** The endpoints of a smooth leaf cannot coincide.

**Pf:** If so there must be a "dead end" singular leaf, as shown.

This is not allowed. $\blacksquare$

So we can pull each smooth leaf in $\tilde{M}$ tight to a geodesic with the same endpoints.

**FACT:** Distinct leaves pull tight to disjoint geodesics.

This defines a $\pi_1$-equivariant map tight: smooth leaves $\rightarrow$ geodesics in $\tilde{M}$.

The image of tight($\mathcal{F}$) is a $\pi_1$-invariant geodesic lamination $\Lambda$ of $\tilde{M}$. For arc $a$ transverse to $\Lambda$ define the measure $\mu(a)$ as the measure of $\text{tight}^{-1}(a \cap \Lambda)$. This defines a $\pi_1$-invariant measured lamination on $\tilde{M}$ that descends to a measured lamination on $M$. 
