Lecture 2 at HUJI 2009-03-25

Recall the def'n of Teichmüller space. Fix a closed oriented surface $M$ of genus $g > 1$. (We'll draw $M$ with genus 2.) Define the set of pairs $\{(X, m)\}$ where $X$ is a hyperbolic surface and $m : M \to X$ is a ~~homeomorphism~~ homeomorphism. ("$m$" stands for "marking.") Define the equivalence relation $\sim$:

$$(X, m_X) \sim (Y, m_Y) \iff \left( \begin{array}{l} \exists \text{ isometry } f : X \to Y \\ \text{s.t. } f \circ m_X \text{ is homotopic to } m_Y. \end{array} \right)$$

~~Can replace homotopic with isotopic without changing $\sim$.~~

Then $\mathcal{T}(M) := \{(X, m)\} / \sim$. So far, this is only a set. Define a metric

$$d((X, m_X), (Y, m_Y)) := \inf \left\{ \log K \;\middle|\; \begin{array}{l} \exists \text{ K-bilipschitz homeom. } f : X \to Y \\ \text{such that } m_Y \underset{\text{homotopic}}{\sim} f \circ m_X \end{array} \right\}$$

on $\mathcal{T}(M)$, turning it into a metric space. (This metric is not particularly interesting, but it easy to define. All the well-known metrics on $\mathcal{T}(M)$ define the same topology.)

FACT: $\mathcal{T}(M) \overset{\text{homeo.}}{=} \mathbb{R}^{6g-6}$ (due to maybe Teichmüller or Bers?)

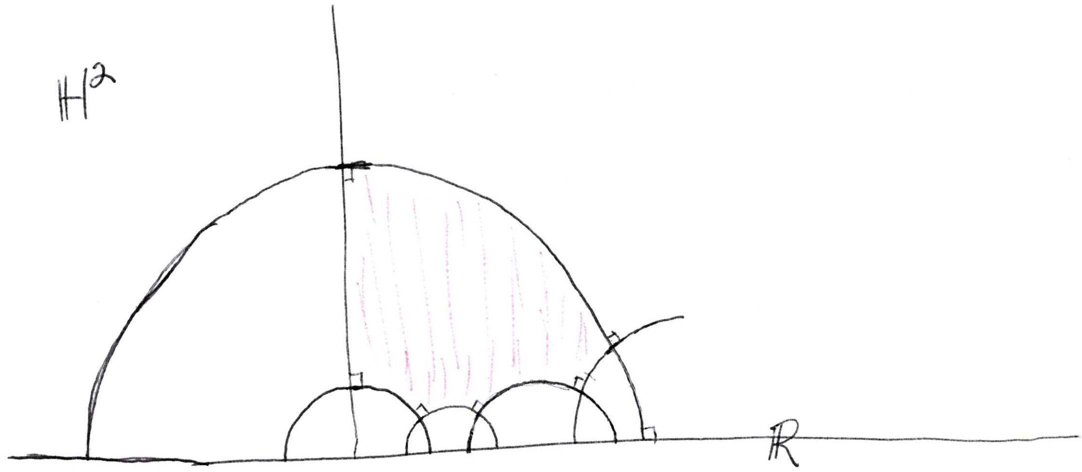
$\mathrm{Mod}(M) \curvearrowright \mathcal{T}(M)$ by isometries via:

$$\varphi \cdot (X, m_X) := (X, m_X \circ \varphi^{-1}).$$

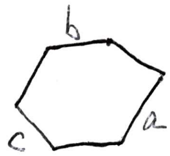
Note this is a mapping class while this is a homeo.
Check this is well defined!

2009-03-25

How can we build a hyperbolic surface? Begin with some planar (hyperbolic) geometry. $\exists$ right-angled hexagons in $\mathbb{H}^2$, e.g.

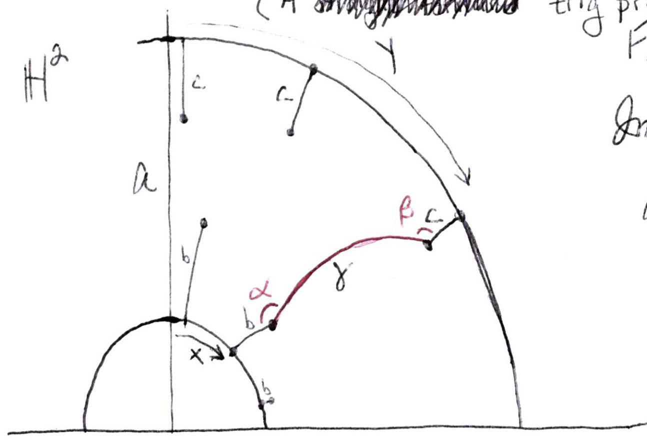



Prop: Label the edges of a hexagon as shown:

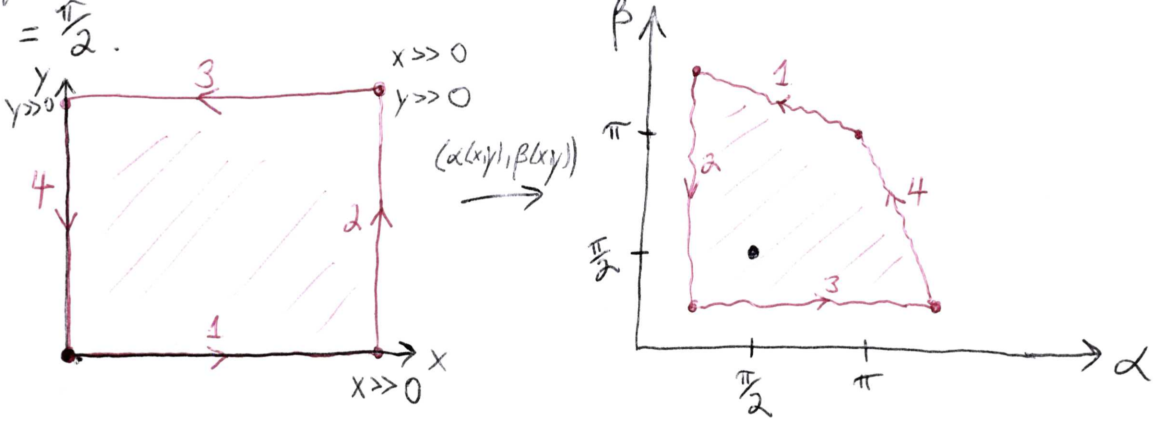



~~[illegible]~~ Pick $l_a, l_b, l_c > 0$. $\exists!$ right-angled hyper. metric on the hexagon such that edge $*$ has length $l_*$ (for $* \in \{a, b, c\}$).

Pf sketch: (This proof is "Thurston-esque".) Wolog $a > b > c$.
(A ~~[illegible]~~ trig proof is possible. See Ratcliffe Thm 3.5.14.)



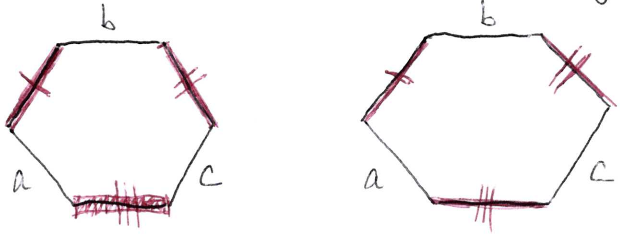

Fix a as a vertical geodesic in $\mathbb{H}^2$. Imagine b & c swinging along geodesics as shown. Let x be the distance from a to b. " y " " " " a " c. Given x & y, let $\gamma$ be the geod. from b to c. Label the angles of $\gamma$ as shown. The goal is to find x & y s.t. $\alpha = \beta = \frac{\pi}{2}$.



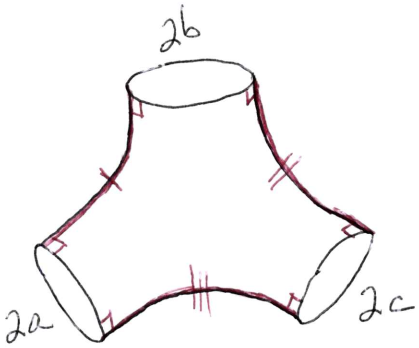
Examine the behavior of $\alpha$ & $\beta$ when at least one of $x, y$ is very large. This is shown in the picture. By continuity $\exists$ values of $x$ & $y$ producing $\alpha = \beta = \frac{\pi}{2}$. $\boxtimes$

(Note: This proof sketch does not discuss uniqueness.)

Given two isometric right-angled hyper. hexagons,



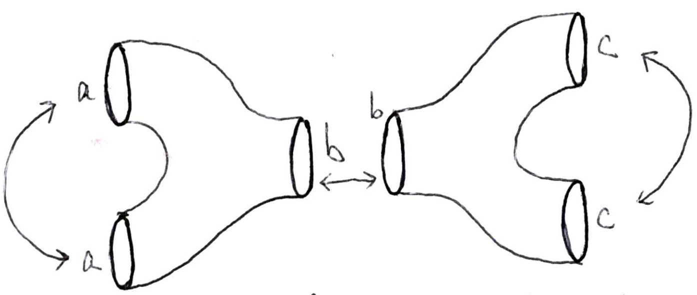

glue along the red edges to produce hyperbolic pants.

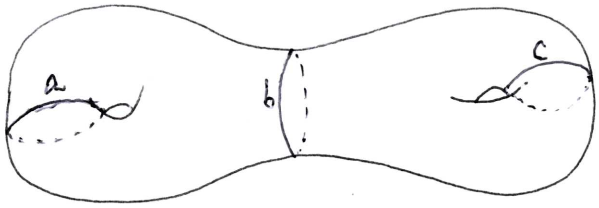



Similarly, given hyperbolic pants with geodesic boundary, add the red geodesics and cut to obtain right-angled hyperbolic hexagons.

Cor: $\forall$ $l_1, l_2, l_3$ $\exists!$ (marked) pants with $\partial$ curves of length $l_1$, $l_2$, & $l_3$.

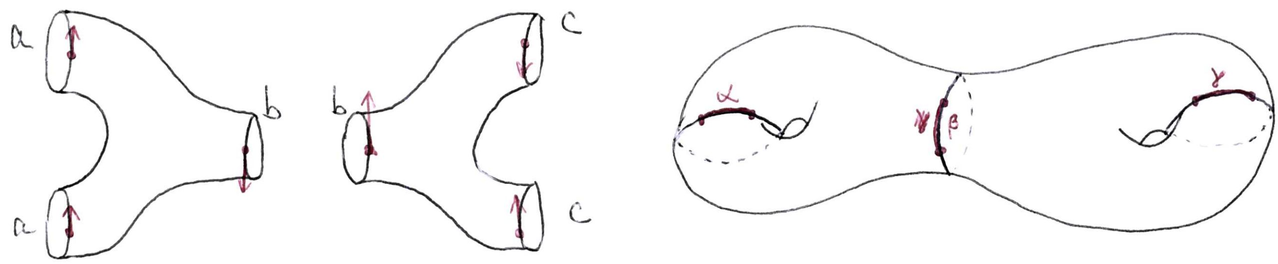
Given two pants with boundary lengths as shown, we can glue to obtain a genus 2 hyper. surface. Similar constructions work in higher genus.




Note there is ambiguity in the gluing.

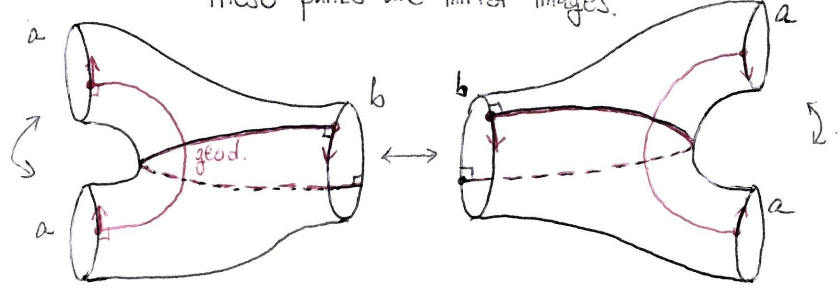
To explicate the ambiguity, add an oriented point to each boundary circle. Then a, b, & c in the closed surface each



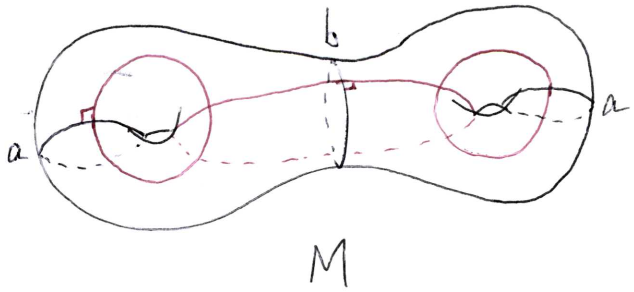

have a well-defined twist $\alpha, \beta, \gamma \in [0, 2\pi)$. Intuitively this indicates that we must specify 6 real parameters $a, b, c, \alpha, \beta, \gamma$ to build a hyperbolic ~~space~~ genus two surface, suggesting the Teichmüller space $\mathcal{T}(\;)$ should have dim'n 6. This is correct, but not a proof. A similar construction in higher genera shows that $\mathcal{T}(M)$ should have dim'n $6g-6$, and this count is correct.

Next I'll describe the Fenchel-Nielsen coord. system on $\mathcal{T}(M)$. For simplicity I'll describe it when M has genus 2. Let's put the following nice hyper. metric on M.

These pants are mirror images.




Red curves are geodesics.




M

As before, for any $l_1, l_2, l_3$ and $\theta_1, \theta_2, \theta_3$ near $0$ we can build a hyperbolic genus two surface $X(l_1, l_2, l_3, \theta_1, \theta_2, \theta_3)$ where $l_1$ is the length of the left curve,

$l_2$ " " " " " middle " ,

$l_3$ " " " " " right " ,

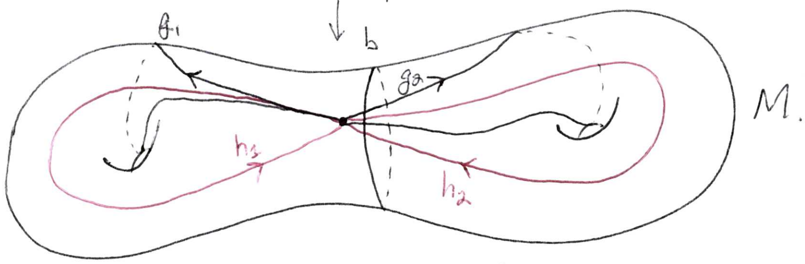
$\theta_1$ is the twist of the left curve,

$\theta_2$ " " " " " middle " , &

$\theta_3$ " " " " " right curve.

Specify a marking $m: M \to X(\vec{l}, \vec{\theta})$ by defining generators for $\pi_1 M$:

The basepoint is to the left of b.



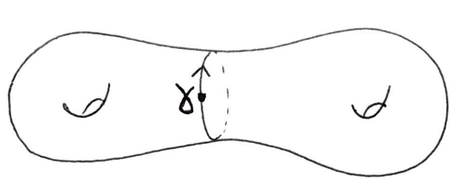

For small $\theta_i$, mark $X(\vec{l}, \vec{\theta})$ by a homeom. taking $g_i$ (on $M$) to $g_i$ (on $X$) & $h_i$ (on $M$) to $h_i$ (on $X$). ~~marking~~ How to extend the for large $\theta_i$? We adjust the marking. Specifically, if $\theta_i \in \mathbb{R}$ build a hyperbolic surface $X(\vec{l}, \vec{\theta})$ using twist parameters ($\theta_i \pmod{2\pi}$) and define a marking taking

$$(g_1 \text{ on } M) \longmapsto (g_1 \text{ on } X(\vec{l}, \vec{\theta}))$$

$$(h_1 \text{ on } M) \longmapsto \left( \begin{array}{l} g_1^k h_1 \text{ on } X(\vec{l}, \vec{\theta}) \text{ where } k \text{ is the} \\ \text{integer floor of } \theta_1/2\pi \text{, i.e. the} \\ \text{greatest int.} \leq \theta_1/2\pi \text{, i.e. } \lfloor \theta_1/2\pi \rfloor \end{array} \right)$$

$$(g_2 \text{ on } M) \longmapsto \left(\gamma^{-k_2} g_2 \gamma^{k_2} \text{ on } X(\vec{l},\vec{\theta}) \text{ where } k_2 = \lfloor \theta_2/2\pi \rfloor\right)$$

$$(h_2 \text{ on } M) \longmapsto \left(\gamma^{-k_2} g_2^{-k_3} h_2 \gamma^{k_2} \text{ on } X(\vec{l},\vec{\theta}) \text{ where } k_3 = \lfloor \theta_3/2\pi \rfloor\right)$$

and $\gamma = g_1 h_1^{-1} g_1^{-1} h_1$ is the curve 

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Abusing notation slightly, let $g_i$ also denote the s.c.c. in the free homotopy class of $g_i$. Then the marking is better described as a ~~the~~ homeomorphism ~~homotopy equivalence~~ with the following action on isotopy classes of scc's.

$$\left.\begin{aligned}
(g_1 \text{ on } M) &\longmapsto (g_1 \text{ on } X(\vec{l},\vec{\theta})) \\
(h_1 \text{ on } M) &\longmapsto \left(D_{g_1}^{k_1} h_1 \text{ for } k_1 = \lfloor \theta_1/2\pi \rfloor\right) \\
(g_2 \text{ on } M) &\longmapsto \left(D_{\gamma}^{+k_2} h_1 \text{ for } k_2 = \lfloor \theta_2/2\pi \rfloor\right) \\
(h_2 \text{ on } M) &\longmapsto \left(D_{\gamma}^{k_2} D_{g_2}^{k_3} h_2 \text{ for } k_3 = \lfloor \theta_3/2\pi \rfloor\right)
\end{aligned}\right\} \quad (\ddagger)$$

Recall $D_*$ is always a right-hand twist along $*$, regardless of any orientation $*$ may have.

This defines a marking $m_{X(\vec{l},\vec{\theta})}$: just apply Dehn twists as prescribed in $(\ddagger)$ to the original identity marking

$$m_{X(a,b,a,0,0,0)} : M \longrightarrow X(a,b,a,0,0,0).$$

So for all $l_1, l_2, l_3 > 0$ & $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ we have defined a point $(X(\vec{l},\vec{\theta}), m_{X(\vec{l},\vec{\theta})})$ in $\mathcal{T}(M)$.

This defines a set map

$$FN: (0,\infty)^3 \times \mathbb{R}^3 \longrightarrow \mathcal{T}(M)$$

Thm (Fenchel - Nielsen): This map is a homeomorphism.

More generally, in higher genus we can define

$$FN: (0,\infty)^{3g-3} \times \mathbb{R}^{3g-3} \longrightarrow \underset{\substack{\uparrow \\ \text{genus } g}}{\mathcal{T}(M)},$$

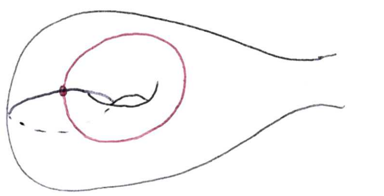
& this map is always a homeom..

(In fact, for any $\varphi \in \mathrm{Mod}(M)$, $(FN^{-1} \circ \varphi \circ FN)$ is a real-analytic diffeom..

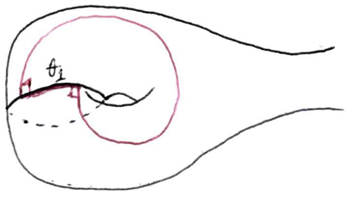
We won't pursue this further.)

This gives an explicit mental picture of $\mathcal{T}(M)$ in terms on "length - twist" coordinates. Let's see an example using the above notation. We look at the left side of our surface as $\theta_1$ increases past $2\pi$. Fix $l_1, l_2, l_3$, & $\theta_2 = \theta_3 = 0$.

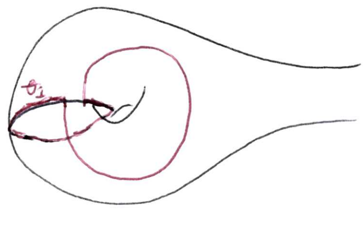
$\theta_1 = 0$



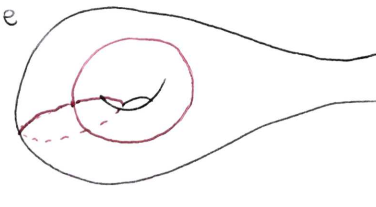
$0 < \theta_1 < 2\pi$



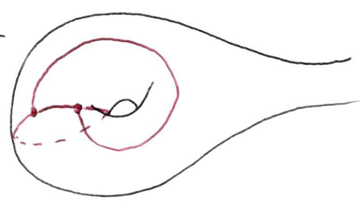

$\theta_1 < 2\pi$ but near $2\pi$




$\theta_1 = 2\pi$, change the marking by $D_{g_1}$.



$4\pi > \theta_1 > 2\pi$, the marking is again $D_{g_1}$ times the original



This completes our description of Fenchel-Nielsen coordinates.

Define $\mathcal{S} = \left\{ \begin{array}{l} \text{isotopy classes of homotopically} \\ \text{nontrivial unoriented} \\ \text{simple closed curves on } M \end{array} \right\}$.

For $c \in \mathcal{S}$ define

$$\ell_c : \mathcal{T}(M) \longrightarrow (0,\infty)$$

$$(X,m) \longmapsto \inf \left\{ \text{length}(c') \;\middle|\; \begin{array}{l} c' \subset X \text{ homotopic} \\ \text{to } m(c) \end{array} \right\}$$

FACT: $\ell_c((X,m))$ is always realized by the length of a simple closed geodesic $c' \subset X$ homotopic to $m(c)$.

Let $(0,\infty)^{\mathcal{S}}$ denote the space of maps $\mathcal{S} \to (0,\infty)$ with the topology of pointwise convergence (aka the product topology). The we have

$$\ell_* : \mathcal{T}(M) \longrightarrow (0,\infty)^{\mathcal{S}}$$

$$(X,m) \longmapsto \{ c \mapsto \ell_c(X,m) \}.$$

Thm (Thurston): $\ell_*$ is a homeomorphism onto its image.

This homeom. is proper.

More is true. Let $\pi : (0,\infty)^{\mathcal{S}} \longrightarrow \mathbb{P}(0,\infty)^{\mathcal{S}}$ denote projectivization.

Thm (Thurston): $\pi \circ \ell_*$ is a homeom. onto its image.

Recall the def'n of a measured singular foliation, $\mathcal{F}$, on $M$.

For a s.c.c. $\alpha \subset M$ define

$$\int_\alpha \mathcal{F} = \sup\left\{ \sum \text{measure}(\alpha_i) \;\middle|\; \begin{array}{l} \alpha_1, \dots, \alpha_\ell \text{ disjoint } \overset{\text{open}}{\wedge} \text{ subarcs of } \alpha \\ \text{transverse to } \mathcal{F} \end{array} \right\}$$

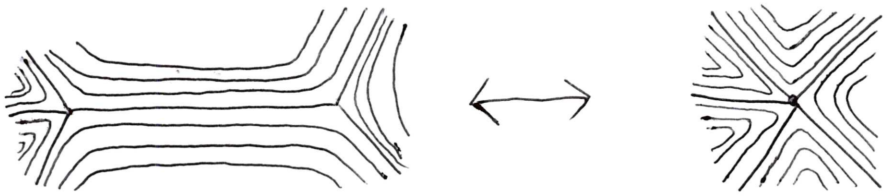
$$= \left(\text{total variation on the measure of } \mathcal{F} \text{ restricted to } \alpha\right).$$

and for $c \in \mathcal{S}$ define $I(\mathcal{F}, c) = \inf\limits_{\alpha \sim c} \int_\alpha \mathcal{F}$.

$I$ stands for intersection. It's possible for $I(\mathcal{F}, c) = 0$, e.g. if $c$ is a closed leaf of $\mathcal{F}$.

Recall $\mathcal{MF}$ is the <u>set</u> of measured singular foliations modulo 2 equivalences:

- isotopy
- Whitehead equivalence



Claim: $I : \mathcal{MF} \times \mathcal{S} \longrightarrow [0, \infty)$ is well-defined.

Taken together, these define a map

$$I_*: \mathcal{MF} \longrightarrow [0, \infty)^{\mathcal{S}}$$

Thm (Thurston): $I_*$ is injective with image disjoint from $\vec{0}$.

Use $I_*$ to define a topology on $\mathcal{MF}$. Let $\mathbb{P}\mathcal{MF}$ denote projective classes of ~~measured singular~~ $\mathcal{MF}$.

Thm (Thurston): $\pi \circ I_* : \mathcal{MF} \to \mathbb{P}[0,\infty)^{\mathcal{S}}$ induces a map

$\mathbb{P}\mathcal{MF} \xrightarrow{I_*} \mathbb{P}[0,\infty)^{\mathcal{S}}$ that is a homeom. onto its image.

Moreover, $I_*(\mathbb{P}\mathcal{MF}) \overset{\text{homeo}}{=} S^{6g-7}$.

Thm (Thurston): Consider $\pi \circ l_*(\mathcal{Y})$, $I_*(\mathbb{P}\mathcal{MF}) \subset \mathbb{P}[0,\infty)^{\mathcal{S}}$.

- $\overline{\pi \circ l_*(\mathcal{Y})} = I_*(\mathbb{P}\mathcal{MF})$
- $(\pi \circ l_*(\mathcal{Y})) \cup (I_*(\mathbb{P}\mathcal{MF}))$ is homeom. to a closed ball.
- Mod(M) acts on this closed ball by homeomorphisms.