All proofs must be written very carefully. Except for the first problem, you may use any results from class or homework. For the first problem you may use anything other than the existence and uniqueness theorem for solutions to ODEs. For the first problem you may use the contraction mapping principle as long as you state clearly how you are using it.

(1) Let $U \subseteq \mathbb{R}^n$ be an open subset. Let $p_0 \in U$. Let $v : U \to \mathbb{R}^n$ be a Lipschitz map. Prove there is a $\delta > 0$ and a continuous path $p : (-\delta, \delta) \to U$ such that $p'(t) = v(p(t))$ for all $t \in (-\delta, \delta)$ and $p(0) = p_0$.

The following problems are true/false. No explanation or proof is required.

(2) (True/False) Let $f(x) = |\sin x|$. There exists a polynomial $p(x)$ such that

$$\sup_{x \in [0,100]} |p(x) - f(x)| < \frac{1}{10}.$$

(3) (True/False) If a metric space is compact then it is also complete.

(4) (True/False) The rational numbers $\mathbb{Q}$ are a countable set.

(5) (True/False) The closed disk $\overline{D_{\ell^2(\mathbb{R})}(0,1)} \subset \ell^2(\mathbb{R})$ is compact.

(6) (True/False) The set

$$\left\{(x, y) \in \mathbb{R}^2 \mid x \in (0,1) \text{ and } y = 0\right\}$$

is an open subset of $\mathbb{R}^2$.

(7) (True/False) If $A \subseteq \mathbb{R}^n$ is a compact subset then $\mathbb{R}^n - A$ is connected.

For the following two problems, let (U_k) be a sequence of open subsets of $\mathbb{R}^n$. The questions are true/false. If the answer is false, *then you must provide a counterexample.*

(8) (True/False) $\cap_{k=1}^{\infty} U_k$ is an open subset of $\mathbb{R}^n$.

(9) (True/False) $\cap_{k=1}^{\infty} (\mathbb{R}^n - U_k)$ is a closed subset of $\mathbb{R}^n$.

(10) Let (f_n) be a sequence of continuous functions $f_n : [0,1] \to [-1,1]$. For each $n \in \mathbb{N}$ and $x \in [0,1]$ define

$$F_n(x) := \int_0^x f_n(t)\, dt.$$

Prove the sequence (F_n) has a subsequence which converges uniformly on $[0,1]$.

(11) For $n \in \mathbb{N}$ define

$$f_n(x) := \left(\frac{\cos(nx)}{n^3}\right) x^5.$$

Prove that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

defines a continuous function on all of $\mathbb{R}$.

(12) Define a function

$$\begin{aligned} F : [0,\pi] &\to \mathbb{R} \\ x &\mapsto \int_0^{x^2} |\cos t|\, dt. \end{aligned}$$

Prove that F is C^1. Write down a formula for F'.

(13) Suppose the metric space (X, d) is connected and has at least two points. Prove that X is not countable. (Hint: you may use the fact that the closed interval $[0, 1]$ is not countable.)