FINAL EXAM

258.31

You may cite anything from Chapters 1-12 of Dummit and Foote, except you may not cite exercises. This test is open book. Any book may be consulted. YOU MAY NOT DISCUSS THIS TEST WITH ANYBODY!! Do not show the test to anybody until after Friday. The exam session on Friday at 10:30 will be closed book, no notes. There are five questions, each worth 20 points.

(1) (20 points)

Fix a field $\mathbb{F}$. For simplicity, assume that the characteristic of $\mathbb{F}$ is not 2. Prove that an $n \times n$ matrix A with entries in $\mathbb{F}$ is invertible if and only if the constant term of its minimal polynomial is different from zero.

(2) (20 points)

Let $p \in \mathbb{N}$ be a prime number. Consider the polynomial

$$f(x) := x^n + px + p^2 \in \mathbb{Z}[x] \quad \text{for} \quad n \geq 4.$$

Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$. (As a first step, prove $f(x)$ has no rational roots.)

(3) (20 points)

For $a_1, a_2, \ldots, a_k \in \mathbb{Z}$, let $(a_1, a_2, \ldots, a_k)$ denote the largest natural number which divides all the integers a_i (for $1 \leq i \leq k$). (In other words, $(a_1, a_2, \ldots, a_k)$ is their greatest common divisor.)

1. Suppose $(a, b, c) = 1$. Prove c can be factored into distinct primes

$$c = \pm p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_\ell^{\alpha_\ell} \cdot q_1^{\beta_1} q_2^{\beta_2} \ldots q_m^{\beta_m} \cdot r_1^{\gamma_1} r_2^{\gamma_2} \ldots r_n^{\gamma_n}$$

(here $\alpha_i, \beta_i, \gamma_i \in \mathbb{N}$) such that

$$\begin{aligned} (a, c) &\quad \text{divides} \quad p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_\ell^{\alpha_\ell} \\ (b, c) &\quad \text{divides} \quad r_1^{\gamma_1} r_2^{\gamma_2} \ldots r_n^{\gamma_n}. \end{aligned}$$

2. Suppose $(a, b, c) = 1$. Prove there exists an integer k such that

$$(a + kb, c) = 1.$$

(4) (20 points)

Make a complete list (with proof) of all the rings satisfying all three of the following properties:

i. R is a commutative ring with multiplicative identity 1 not equal to 0,
ii. R has a unique maximal ideal, and
iii. 1 is the unique unit of R.

(5) (20 points)

Equip $\mathbb{R}^3$ with the basis $\mathcal{B} := \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Exhibit a real 3×3 matrix having minimal polynomial $(t^2 + 2)(t - 7)$, which, as a linear

transformation of $\mathbb{R}^3$ equipped with the basis $\mathcal{B}$,
1. leaves invariant the line L through $(0, 0, 0)$ and $(2, -1, 3)$, and
2. leaves invariant the plane P through $(0, 0, 0)$ perpendicular to L (i.e. P is the plane given by all the points (x, y, z) satisfying $2x - y + 3z = 0$.)