Math 370 (Algebra I)

Test HW (two pages)

Recall that to disprove an assertion means to *give an example* where the assertion is not true.

Example. Prove/disprove that:

a) For all $x, y \in \mathbb{R}$ there exists $z \in \mathbb{R}$ such that $z^2 = x^2 + y^2$.

b) The same question for $x, y, z \in \mathbb{Q}$.

Solution. The assertion a) is true, because for $x, y \in \mathbb{R}$ one has: $x^2, y^2 \geqslant 0$, thus $x^2 + y^2 \geqslant 0$. Further, $\alpha := x^2 + y^2 \geqslant 0$ in $\mathbb{R}$, hence $\exists\, z \in \mathbb{R}$ s.t. $z^2 = \alpha$ (WHY).

The assertion b) is wrong, because $x = 1 = y$ implies that $x^2 + y^2 = 2$, and there is no $z \in \mathbb{Q}$ such that $z^2 = 2$, because $\sqrt{2} \notin \mathbb{Q}$ (WHY).

Basics of logical deduction.

Given assertions $p, q, r, \dots$

Recall the (logical) disjunction/conjuction $p \vee q$, resp. $p \,\&\, q$, and recall the (logical) negation $\neg\, p$, and recall that using parentheses is essential for building desired unambiguous assertions. Further recall the (universal) quantifier $\forall$, the (existential) quantifier $\exists$, and their usage.

1) Answer/prove/disprove the following:

a) The assertion $p \vee q \,\&\, r$ is ambiguous, because the possible interpretations are
(i) $(p \vee q) \,\&\, r$; (ii) $p \vee (q \,\&\, r)$. Prove that (i) and (ii) are not equivalent in general.

b) Prove that: (i) $\neg(p \vee q)$ is the same as $\neg p \,\&\, \neg q$; (ii) $\neg(p \,\&\, q)$ is the same as $\neg p \vee \neg q$.

c) Let $p(x)$ be an assertion depending on x. Then one has:
a) $\neg\,(\forall\, x\ p(x))$ is the same as $\exists\, x\,(\neg\, p(x))$; b) $\neg\,(\exists\, x\ p(x))$ is the same as $\forall\, x\,(\neg\, p(x))$.

2) In the set of natural numbers $\mathbb{N} = \{0, 1, \dots\}$, consider the assertion in plain English:

$(*)$ *Every natural number less than 2022 is a sum of three squares of natural numbers.*

a) What is the negation of the assertion $(*)$ in plain English. Is the assertion $(*)$ true?

b) Write the assertion $(*)$ and its negation using quantifiers.

Sets, correspondences, maps, relations.

Spend some time trying to get used to the axiomatic way of thinking & working with sets!

3) Let A, B, C, D be given sets, and x be elements, e.g., real numbers. Answer the following:

a) Using $\cup$, $\cap$, $\setminus$ and A, B, C, D write down the sets of all x which satisfy:
i) $(\,x \in A$ or $x \in B\,)$ & $x \in C$ & $x \notin D$; ii) $x \in A$ or $(\,x \in B$ & $x \in C\,)$ & $x \notin D$.

b) Write as a union of disjoint intervals the sets of the real numbers $x \in \mathbb{R}$ satisfying:
i) $(\,x < 20$ & $x^2 < 100\,)$ or $x \notin (-\infty, -1]$; ii) $x < 20$ & $(\,x^2 < 100$ or $x \notin (-\infty, -1]\,)$.

$(*)$ Does the place of the parentheses matter?

4) Let A, B, X, Y, X', Y' be sets. Using the ZF axioms, prove:

(i) $s(A) := A \cup \{A\}$ is a set, called the successor (set) of A.

(ii) $(X, Y) := \{\{X\}, \{X, Y\}\}$ is a set called the ordered pair with entries X, Y.

Further, prove that the following hold:

a) $s(A) = s(B)$ iff $A = B$. Further, $(X, Y) = (X', Y')$ iff $X = X'$ and $Y = Y'$.

b) $A \times B := \{(X, Y) \,|\, X \in A, Y \in B\}$ is a set, called the (Cartesion) product of A, B.

5) Using the definitions of $\cup$, $\cap$, $\setminus$, $\times$, prove the following:

a) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

b) *de Morgan* laws: $C\setminus(A\cup B) = (C\setminus A)\cap(C\setminus B)$, $C\setminus(A\cap B) = C\setminus A\cup C\setminus B$.

c) $(A\cup B)\times C = (A\times C)\cup(B\times C)$ and $(A\cap B)\times C = (A\times C)\cap(B\times C)$.

Correspondences / maps. Recall (functional) correspondences $R\subset A\times B$, etc.

6) Let $f : A\to B$, $g : B\to C$ be maps, and consider $g\circ f : A\to C$. Prove the following:

a) If f and g are injective (reps. surjective), then $g\circ f$ is injective (reps. surjective). Is the converse assertion true ? (What is that?)

b) If f and g are bijective, then $g\circ f$ is bijective. Is the converse assertion true?

7) Using the ZFC system of axioms, prove/answer the following:

a) There is $f : A\to B$ injective iff there is $g : B\to A$ surjective.

b) $f : A\to B$ is injective iff $\exists\, g : B\to A$ surjective satisfying $g(f(x)) = x\ \forall\, x\in A$.

• **Recall**: Let $X\xrightarrow{h}Y$ be an arbitrary map. Then for $A\subset X$ and $B\subset Y$ one has:

a) $f(A) := \{y\in Y \mid \exists\, x\in A \text{ s.t. } f(x) = y\}$ is a subset of Y (WHY), the image of A under f.

b) $f^{-1}(B) := \{x\in X \mid f(x)\in B\}$ is a subset of X (WHY), the pre-image of B under f.

9) Prove/disprove that for all subsets $A', A''\subset X$ and $B', B''\subset Y$ one has:

a) $f(A'\cap A'') = f(A')\cap f(A'')$, respectively $f(A'\cup A'') = f(A')\cup f(A'')$.

b) $f^{-1}(B'\cap B'') = f^{-1}(B')\cap f^{-1}(B'')$, respectively $f^{-1}(B'\cup B'') = f^{-1}(B')\cup f^{-1}(B'')$.

(•) The same questions provided f is injective, resp. surjective, resp. bijective.

Cardinality of sets. Let $|A|$ be the cardinality of A, and say $|A|\leqslant|B| \overset{\text{Def}}{\longleftrightarrow} \exists\, f : A\to B$ injective. Recall that X is called finite of cardinality $|X| = n$, if either $X = \emptyset$ and then $|X| = 0$, or there is a bijection $f : \{1,\dots,n\}\to X$, thus $X = \{x_i := f(i) \mid i = 1,\dots,n\} = \{x_1,\dots,x_n\}$. Check the basic facts concerning finite sets in the notes Math ABC.

10) Let X be an arbitrary set, and $\mathcal{P}(X) := \{A \mid A\subseteq X\}$ be the power set of X. Prove:

a) If $|X| = n$ is finite, then $|\mathcal{P}(X)| = 2^n$.

b) One always has that $|X| < |\mathcal{P}(X)|$. Deduce from this that $|\mathbb{N}| < |\mathbb{R}|$.

[**Hint** to the second part of b): Define $f : \mathcal{P}(\mathbb{N})\to\mathbb{R}$ by $f(A) := \overline{a_0.a_1\dots a_n\dots}$ for $A\subseteq N$, where $a_n = 1$ if $n\in N$, and $a_n = 0$ if $n\notin N$. Then $N\neq N'$ implies $x_N\neq x_{N'}$ (WHY), hence f is injective, etc...]

11) Let X, Y be finite sets, say $|X| = m$ and $|Y| = n$. Prove/disprove the following:

a) $|X\cup Y| + |X\cap Y| = |X| + |Y|$. What is the corresponding assertion for $|X\cup Y\cup Z|$?

b) $|X\times Y| = |X|\cdot|Y|$. What is the corresponding assertion for $|X\times Y\times Z|$?

[**Hint**: Proof by induction on m, n, etc.]

Proofs by contradiction / Direct proofs.

12) Prove that $\sqrt[3]{5}$ is not a rational number. Same qustion for $e = \sum_n \frac{1}{n!}$ (What does that mean?)

13) Let $\emptyset\neq X\subset\mathbb{Q}$, $X\neq\{0\}$ be closed w.r.t. subtraction and division, i.e., $\forall\, x, y\in X, y\neq 0$ one has: $x-y, x/y\in X$. Prove that $X = \mathbb{Q}$. What is the corresponding assertion for $X\subset\mathbb{Z}$?

14) In the set of real numbers $\mathbb{R}$ consider the assertion: $(*)\ \forall\, x, y\in\mathbb{R}\ \exists\, z\in\mathbb{R}$ s.t. $x^2 + y^2 = z^2$

a) Formulate $(*)$ and its negation in plain English, and write the negation of $(*)$ with quantifiers.

b) Prove that the implication above is true, both directly, and arguing by contradiction.