Due: April 12, 2024

Math 6030 / Problem Set 9 (two pages)

More about valuation rings

Let R be a UFD and $\mathcal{P} \subset R$ be a set of representatives for the prime elements modulo association, i.e., $\pi \sim \pi' \overset{\text{def}}{\Longleftrightarrow} \pi R^\times = \pi' R^\times$. Recall that every $r \in R$ has a unique presentation of the form $r = \epsilon_r \prod_{\pi \in \mathcal{P}} \pi^{n_{r,\pi}}$ with $n_{r,\pi} \in \mathbb{N}$ and $n_{r,\pi} = 0$ for almost all (for short, f.a.a.) $\pi \in \mathcal{P}$ (WHY), and every $x = \frac{a}{r} \in K = \mathrm{Quot}(R)$ has a unique presentation of the form $x = \epsilon_x \prod_{\pi \in \mathcal{P}} \pi^{n_{x,\pi}}$ with $n_{r,\pi} \in \mathbb{Z}$ and $n_{r,\pi} = 0$ f.a.a (for almost all) $\pi \in \mathcal{P}$ (WHY).

1) In the above notation, consider the map $v_\pi : K \to \mathbb{Z} \cup \infty$ defined by $v_\pi(x) = n_{x,\pi}$ if $x \neq 0_K$ and $v_\pi(0_K) = \infty$. Prove/disprove/answer:

a) v_π is a discrete valuation, which does not depend on π, but rather on $\pi R^\times$.

b) What is the valuation ring $R_{v_\pi}, \mathfrak{m}_{v_\pi}$, its units $R^\times_{v_\pi}$, and the residue field κ_{v_π}?

c) Are all the discrete valuation rings R_v with $R \subset R_v$ of the form $R_v = R_{v_\pi}$?

Modules over PIDs

Recall that a torsion R-module M is called π-primary (torsion module), if M is π^∞-torsion, i.e., for every $x \in M$ there is $n > 0$ such that $\pi^n x = 0_M$.

2) In the notation above, suppose that R is a PID, and M is a finite torsion R-module. Prove/disprove/answer the following:

a) For each $\pi \in \mathcal{P}$ there is a unique π-primary R-submodule $M_{(\pi)} \subset R$ s.t. $R_{(\pi)} = (0)$ f.a.a. $\pi \in \mathcal{P}$ and $M = \oplus_\pi M_{(\pi)}$. **Terminology**. $M_{(\pi)}$ is the π-primary component of M.

b) For every $M_{(\pi)} \neq (0)$ there are unique $0 < n_1 \leqslant \cdots \leqslant n_r = n_{r_\pi}$ s.t. $M_{(\pi)} \cong \oplus_i R/(\pi^{n_i})$. What can you say about $\pi^{n_1}, \ldots, \pi^{n_r}$?

3) Given $A = \left(\begin{smallmatrix} 6 & 3 \\ 2 & 3 \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$, $A_t := tI_2 - A \in \mathbb{Q}[t]^{2\times 2}$, and $\mathcal{E} = (e_1, e_2)$, define morphisms by:

$$\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2,\ \varphi(\mathcal{E}) \mapsto (x_1, x_2) := \mathcal{E}A, \quad \varphi_t : \mathbb{Q}[t]^2 \to \mathbb{Q}[t]^2,\ \varphi_t(\mathcal{E}) \mapsto (y_1, y_2) := \mathcal{E}A_t.$$

Find bases $\mathcal{B} = (\alpha_1, \alpha_2)$ of M and $\delta_1 | \delta_2$ s.t. $\mathcal{B} = (\delta_1\alpha_1, \delta_2\alpha_2)$ are basis of $N \subset M$ in the cases:

a) $M := \mathbb{Z}^2$ and $N = \varphi(M) \subset M$.

b) $M := \mathbb{Q}[t]^2$ and $N = \varphi_t(M) \subset M$.

4) Find the invariant factors of the matrix $A \in \mathbb{C}^{n\times n}$ in the cases:

$$\text{a)}\ A = \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix} \qquad \text{b)}\ A = \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ i & 0 & 1 \end{pmatrix} \qquad \text{c)}\ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

[**Hint:** Using elementary matrices over $\mathbb{C}[t]$, transform A_t in the diagonal form with $\delta_1 | \ldots | \delta_n$ on diagonal, etc...]

5) Let R be a Euclidean domain w.r.t. $\varphi : R \to \mathbb{N}$, and $N \subset M = R^n$ be generated by $\mathcal{X} = (x_i)_i$, $x_i = (a_{i1}, \ldots, a_{in}) \in R^n$ for $i = 1, \ldots, m$. Evaluate the number of necessary multiplications in terms of $\|\mathcal{X}\| := \max_{i,j} \varphi(a_{ij})$ in order to find a basis $\mathcal{A} = (\alpha_1, \ldots, \alpha_n)$ of M and $\delta_1 | \ldots | \delta_n$ in R s.t. N is generated by $\mathcal{B} = (\delta_1\alpha_1, \ldots, \delta_n\alpha_n)$.

ACC/DCC. In the sequel, R is a (not necessarily commutative) ring with 1_R, and recall the notation/convention form the class: $\bullet$ denotes l(left), r(right), bi(left&right), and we speak about the set $\mathcal{M}_\bullet$ of $\bullet$-R-submodules of an $\bullet$-R-module M, e.g. the set of $\bullet$-ideals $\mathcal{I}d_\bullet(R)$ of R. Recall that an increasing/decreasing (w.r.t. $\subset$) sequence $(N_i)_i$ in $\mathcal{M}_\bullet$ satisfies ACC/DCC if the sequence is stationary, i.e., $\exists\, i_0$ such that $N_i = N_{i_0}$ for $N_{i_0} \subset N_i$, resp. $N_i \subset N_{i_0}$, and M satisfies ACC/DCC is all increasing/decreasing sequences in $\mathcal{M}_\bullet$ satisfy ACC/DCC. Finally, M is $\bullet$-Noetherian if $\mathcal{M}_\bullet$ satisfies ACC, respectively $\bullet$-Artinian if $\mathcal{M}_\bullet$ satisfies DCC.

6) Prove the assertions for the class:

a) M satisfies ACC/DCC iff all subsets $\emptyset \neq \mathcal{X} \subset \mathcal{M}_\bullet$ have maximal/minimal elements.

b) M satisfies ACC iff every $N \in \mathcal{M}_\bullet$ is finitely generated.

c) Let $\mathfrak{a} \in \mathcal{I}d_\bullet(R)$ be given. If M satisfies ACC/DCC, then one has:

(l) $M/\mathfrak{a}M$; (r) $M/M\mathfrak{a}$; (bi) $M/\mathfrak{a}M$ & $M/M\mathfrak{a}$ satisfy ACC/DCC.

7) Prove the assertions for the class/answer:

a) Let R be commutative, $\Sigma \subset R$ is a multiplicative system, and M satisfies ACC/DCC. Then the R_Σ-module M_Σ satisfies ACC/DCC.

b) If R is a skew field and V is an $\bullet$-R-vector space, TFAE:

(i) V satisfies ACC; (ii) V satisfies DCC; (iii) V is finite dimensional.

8) Let $0 \to M_0 \to \cdots \to M_n \to 0$ be an exact sequence of $\bullet$-R-modules. Prove that all the modules $(M_{2k})_{k\geqslant 0}$ satisfy ACC/DCC iff all the modules $(M_{2k+1})_{k\geqslant 0}$ satisfy ACC/DCC.

Composition series

Recall that an $\bullet$-R-module M has a composition series iff M satisfies both ACC and DCC. Further, by the Jodan–Hölder Thm, all non-redundant $\bullet$-composition series have the same length, denoted $\ell(M) \in \mathbb{N}$, and the simple factors are isomorphic up to a permutation (WHY).

[Make sure that you review/know the proof(!)]

9) Let $R^{\mathsf{ACC}}_{\mathsf{DCC}}$-**Mod** be the category of $\bullet$-R-modules satisfying ACC & ADD. Prove/disprove/answer:

a) $R^{\mathsf{ACC}}_{\mathsf{DCC}}$-**Mod** is closed w.r.t. taking $\bullet$-R factor and submodules, finite products/coproducts.

b) $\ell : R^{\mathsf{ACC}}_{\mathsf{DCC}}$-**Mod** $\to \mathbb{N}$ is additive, i.e., $0 \to M' \to M \to M'' \to 0$ exact, then $\ell(M) = \ell(M') + \ell(M'')$.

c) If $0 \to M_1 \to \ldots \to M_n \to 0$ is an exact sequence in $R^{\mathsf{ACC}}_{\mathsf{DCC}}$-**Mod**, then $\sum_i (-1)^i \ell(M_i) = 0$.