Due: March 18, 2024

Math 6030 / Problem Set 6 (two pages)

More about Hom_R and Exactness (continued)

Recall that an R-module Q is called injective, if Q has the *colifting property,* i.e., if for any *injective* morphism $f : M' \to M$ of R-modules, $\mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M', Q)$ is surjective. Make sure that you know/checked the characterizations of Q being injective, e.g. Q injective iff $\mathcal{H}^Q = \mathrm{Hom}_R(\bullet, Q)$ is exact iff Q is direct summand in every R-module containing Q iff Q satisfies Baer's criterion ...

1) Let M, M_i be R-modules, S be a commutative R-algebra. Prove/disprove/answer:

a) Q is injective iff $Q_{\mathfrak{p}}$ is injective $\forall\, \mathfrak{p} \in \mathrm{Spec}(R)$ iff $M_{\mathfrak{m}}$ is injective $\forall\, \mathfrak{m} \in \mathrm{Max}(R)$.

b) $M = \prod_\alpha M_\alpha$ is injective iff all M_α are injective. Does the same hold for $M = \oplus_\alpha M_\alpha$?

c) If M is a injective, then M is flat. Does the converse hold ?

2) Let S be a commutative R-algebra. Prove the assertion from class:

a) If Q is and injective S-module, then Q is (by restriction of scalars) injective over R.

b) If Q is an injective R-module, the co-induced S-module $M^S := \mathrm{Hom}_R(S, M)$ is injective.

3) Prove the assertions form class:

a) An abelian group A is an injective $\mathbb{Z}$-modules iff A is divisible, e.g., $\mathbb{Q}/\mathbb{Z}, +$ is so.

b) If R-**Mod** $\rightsquigarrow$ R-**Mod**, $M \mapsto M^D := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a contravariant exact functor.

c) $M \to M^{DD} := (M^D)^D$, $x \mapsto \varphi(x)\ \forall\, \varphi \in M^D$ is an injective morphism of R-modules.

Resolutions of R-modules /// Tor_i^R and Ext_R^i

Recall that given an R-module M, a projective/free/flat resolution of M is any *exact sequence* of the form $(\mathcal{P}_M) : \ldots \to P_1 \to P_0 \to M \to 0$, where $P_0, P_1, \ldots$ are projective/free/flat R-modules. Similarly, an *exact sequence* $(\mathcal{Q}_N) : 0 \to M \to Q_0 \to Q_1 \to \ldots$ with $Q_0, Q_2, \ldots$ injective R-modules is called an injective resolution of a given R-module N. **Notice** that both multiple projective/free/flat resolutions $(\mathcal{P}_M)$ and/or injective resolutions $(\mathcal{Q}_N)$ exist (WHY). Finally, resolutions $\mathcal{P}_M$ and/or $\mathcal{Q}_N$ as above, consider the resulting sequences in R-**Mod**:

a) $\mathcal{P}_M \otimes_R N$: $\quad \ldots \longrightarrow P_2 \otimes_R N \xrightarrow{\pi_2} P_1 \otimes_R N \xrightarrow{\pi_1} P_0 \otimes N \xrightarrow{\pi_0} 0$

b) $\mathrm{Hom}_R(\mathcal{P}_M, N)$: $\quad 0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{p_0} \mathrm{Hom}_R(P_1, N) \xrightarrow{p_1} \mathrm{Hom}_R(P_2, N) \xrightarrow{p_2} \ldots$

c) $\mathrm{Hom}_R(M, \mathcal{Q}_N)$: $\quad 0 \to \mathrm{Hom}_R(Q_0, N) \xrightarrow{q_0} \mathrm{Hom}_R(Q_1, N) \xrightarrow{q_1} \mathrm{Hom}_R(Q_2, N) \xrightarrow{q_2} \ldots$

4) Prove the above sequences of R-modules at a), b), c) above are complexes.

Recall: The *homology groups* of the complex $\mathcal{P}_M \otimes_R N$ are called the Tor-groups of M, N, denoted $\mathrm{Tor}_i^R(M, N) := \mathrm{Ker}(\pi_i)/\,\mathrm{Im}(\pi_{i+1})$. The *cohomology groups* of the complex $\mathrm{Hom}_R(\mathcal{P}_M, N)$ are called the Ext-groups of M, N, denoted $\mathrm{Ext}_R^i(M, N) := \mathrm{Ker}(p_i)/\,\mathrm{Im}(p_{i-1})$. The following fundamental facts hold (try to study the proofs, which are a little bit technical!):

Thm (Baer; Eilenberg, MacLane). *$\mathrm{Tor}_i^R(M, N)$ and $\mathrm{Ext}_R^i(M, N)$ are <u>independent</u> of the resolutions used to compute them, and $\mathrm{Ext}_R^i(M, N)$ equal the cohomology groups of $\mathrm{Hom}_R(M, \mathcal{Q}_N)$, that is,*

$$\mathrm{Ext}_R^i(M, N) = \mathrm{Ker}(q_i)/\,\mathrm{Im}(q_{i-1}).$$

- The following (quite obvious) properties of Tor^R and Ext_R hold (try to prove!):
  - a) $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$ and $\mathrm{Ext}_R^0(M,N) = \mathrm{Hom}_R(M,N)$ (WHY).
  - b) If M or N is flat, $\mathrm{Tor}_1^R(M,N) = (0)$ (WHY). What about $\mathrm{Tor}_i^R(M,N)$?
  - c) If M or N is projective/injective, $\mathrm{Ext}_R^1(M,N) = (0)$. What about $\mathrm{Ext}_R^i(M,N)$?
  - d) $\mathrm{Tor}_i^R(\oplus_\alpha M_\alpha, N) = \oplus_\alpha \mathrm{Tor}_i^R(M_\alpha, N)$ and $\mathrm{Tor}_i^R(\varinjlim_\alpha M_\alpha, N) = \varinjlim_\alpha \mathrm{Tor}_i^R(M_\alpha, N)$.
  - e) $\mathrm{Ext}_R^i(\oplus_\alpha M_\alpha, N) = \prod_\alpha \mathrm{Ext}_R^i(M_\alpha, N)$ and $\mathrm{Ext}_i^R(M, \prod_\alpha N_\alpha) = \prod_\alpha \mathrm{Ext}_R^i(M, N_\alpha)$.

5) Prove/disprove:

a) Tor_i^R and Ext_R^i are compatible with taking rings of fractions R-**Mod** $\rightsquigarrow R_\Sigma$-**Mod**, i.e., $\big(\mathrm{Tor}_i^R(M,N)\big)_\Sigma = \mathrm{Tor}^{R_\Sigma}(M_\Sigma, N_\Sigma)$, and $\big(\mathrm{Ext}_R^i(M,N)\big)_\Sigma = \mathrm{Ext}_{R_\Sigma}^i(M_\Sigma, N_\Sigma)$.

b) What about the behavior of Tor_i^R and Ext_R^i under localization?

- Finally, given short exact sequences $0 \to M' \to M \to M'' \to 0$, $0 \to N' \to N \to N'' \to 0$, Tor^R and Ext_R give rise to long exact sequences as follows (check the proof!):

$$\cdots \to \mathrm{Tor}_2^R(M'',N) \to \mathrm{Tor}_1^R(M',N) \to \mathrm{Tor}_1^R(M,N) \to \mathrm{Tor}_1^R(M'',N) \to \mathrm{Tor}_0^R(M',N) \to \mathrm{Tor}_0^R(M,N) \to \mathrm{Tor}_0^R(M'',N) \to 0$$

$$0 \to \mathrm{Ext}_R^0(N'',M) \to \mathrm{Ext}_R^0(N,M) \to \mathrm{Ext}_R^0(N',M) \to \mathrm{Ext}_R^1(N'',M) \to \mathrm{Ext}_R^1(N,M) \to \mathrm{Ext}_R^1(N',M) \to \mathrm{Ext}_R^2(N',M) \to \dots$$

$$0 \to \mathrm{Ext}_R^0(M,N') \to \mathrm{Ext}_R^0(M,N) \to \mathrm{Ext}_R^0(M,N'') \to \mathrm{Ext}_R^1(M,N') \to \mathrm{Ext}_R^1(M,N) \to \mathrm{Ext}_R^1(M,N'') \to \mathrm{Ext}_R^2(M,N') \to \dots$$

6) For $r \in R$ and an R-module N, we denote ${}_rN := \{x \in N \mid rx = 0_N\}$.
Supposing that $r \in R$ is not a zero-divisor, prove/disprove:

a) $\mathrm{Tor}_0^R(R/rR, N) = N/rN$; $\mathrm{Tor}_1^R(R/rR, N) = {}_rN$; $\mathrm{Tor}_i^R(R/rR, N) = (0)$ for $i > 1$.

b) $\mathrm{Ext}_R^0(R/rR, N) = {}_rN$; $\mathrm{Ext}_R^1(R/rR, N) = N/rN$; $\mathrm{Ext}_R^i(R/rR, N) = (0)$ for $i > 1$.

Make educated guesses: *What should be the above assertions for* $r_1, \dots, r_n \in R$, $n = 2, \dots$.

[**Hint**: $0 \to rR \to R \to R/rR \to 0$ is a projective resolution of $M := R/rR$ (WHY), $0 \to {}_rN \to N \to N/{}_rN \to 0$ is exact, etc. . . .]

- Recall that given a group G acting on an abelian group A, we defined the cohomology groups $H^i(G,A) = Z^i(G,A)/B^i(G,A)$ for $i = 1, 2$. Supposing that G, A are R-modules and G acts trivially on A, consider $B^i_{R\text{-}\mathbf{Mod}}(G,A) \subset Z^i_{R\text{-}\mathbf{Mod}}(G,A)$ as R-modules (HOW), hence $H^i_{R\text{-}\mathbf{Mod}}(G,A)$ are R-modules (WHY).

7) In the above notation, prove/disprove: $H^i_{R\text{-}\mathbf{Mod}}(G,A) = \mathrm{Ext}_R^{i-1}(G,A)$ for $i = 1, 2$.