Due: Feb 26, 2024

Math 6030 / Problem Set 4 (two pages)

Dual/bidual of an R-module. Let R be a commutative ring, M be an R-module.

• $\mathrm{Maps}(X,M)$ is the R-module of M-valued maps on X. The support of $f \in \mathrm{Maps}(X,M)$ is $\sup(f) := \{x \in X \mid f(x) \neq 0_M\}$, and $\mathrm{Maps}^0(X,M) := \{f \in \mathrm{Maps}(X,M) \mid \sup(f) \text{ finite}\}$ is an R-submodule of $\mathrm{Maps}(X,M)$ (WHY).

• **Recall:** The dual of M is $M^\vee := \mathrm{Hom}_R(M,R)$, and the bidual of M is $M^{\vee\vee} := (M^\vee)^\vee$ the dual of $M^\vee$. The map $\imath : M \to M^{\vee\vee}$, $x \mapsto \Psi_x$, $\Psi_x(\phi) := \phi(x)$ for $\phi \in M^\vee$ is the evaluation morphism, and M is called reflexive, if $\imath : M \to M^{\vee\vee}$ is an isomorphism of R-modules.

• Finally recall the Kronecker symbol $\delta_{ij} \in \{0_R, 1_R\}$ defined for i,j in an arbitrary set.

1) Suppose that M is free, with basis $\mathcal{A} = (\alpha_i)_{i\in I}$. Prove/disprove/answer:

a) The maps $\alpha_i^\vee : M \to R$ define by $\alpha_j \mapsto \delta_{ji}$ are R-linearly independent maps.

b) If I is finite, then $\mathcal{A}^\vee := (\alpha_i^\vee)_i \in I$ is an R-basis of $M^\vee$ such that $\imath(\mathcal{A}) = \mathcal{A}^{\vee\vee}$.
In particular, $\imath : M \to M^{\vee\vee}$ is an isomorphism, hence $M \cong_R M^\vee$ (non-canonically).

c) If I is infinite, then $\imath : M \to M^{\vee\vee}$ is injective, but never surjective.

2) Prove that R-**Mod** $\rightsquigarrow$ R-**Mod**, $M \mapsto M^\vee$ is a contravariant functor. Further, prove/disprove:

a) The functor above maps free R-modules to free R-modules.

b) Let N, M be free with bases $\mathcal{B} = (\beta_j)_{1\leqslant j\leqslant n}$, $\mathcal{A} = (\alpha_1, \dots, \alpha_m)$, and $f \in \mathrm{Hom}_R(N,M)$ have matrix $A_f \in R^{m\times n}$ in the bases $\mathcal{B}, \mathcal{A}$, i.e., $f(\mathcal{B}) = \mathcal{A}\cdot A_f$. Then the matrix of $f^\vee : M^\vee \to N^\vee$ in the dual bases $\mathcal{A}^\vee$, $\mathcal{B}^\vee$ is $A_{f^\vee} = A_f^\tau \in R^{n\times m}$, i.e., $f^\vee(\mathcal{A}^\vee) = \mathcal{B}^\vee \cdot A_f^\tau$.

Bilinear/multiliear maps & Tensor product. For $(M_\nu)_{\nu\leqslant n}, N$ in R-**Mod**, denote by $\mathcal{L}(\prod_\nu M_\nu, N)$ the set of R-multilinear maps $f : \prod_\nu M_\nu \to N$, set $\mathcal{L}^n(M;N) := \mathcal{L}(M^n, N)$, and let $\mathcal{L}^n_{\text{sym}}(M;N), \mathcal{L}^n_{\text{alt}}(M;N) \subset \mathcal{L}^n(M;N)$ are the sets of the symmetric, resp. alternating maps. Make sure that you *check all the details* proving that $\mathcal{L}(\prod_\nu M_\nu, N) \subset \mathrm{Maps}(\prod_\nu M_\nu, N)$ and $\mathcal{L}^n_{\text{sym}}(M;N), \mathcal{L}^n_{\text{alt}}(M;N) \subset \mathcal{L}^n(M;N)$ are R-submodules. Further, let $\mathcal{X}_\nu, \mathcal{A}_\nu$, respectively $\mathcal{X} = (x_i)_i, \mathcal{A} = (\alpha)_i$, $i \in I$ be a system of generators/R-basis of each M_ν, respectively of M. I^n acts on, hence on $\mathcal{X}^n$ and $\mathcal{A}^n$ by $\sigma(i_1,\dots,i_n) = (i_{\sigma 1},\dots,i_{\sigma n})$, and let $\mathcal{X}^n_\leqslant$, $\mathcal{A}^n_\leqslant$, respectively $\mathcal{X}^n_<$, $\mathcal{A}^n_<$ be systems of representatives for the S_n action $\mathcal{X}^n$, $\mathcal{A}^n$, respectively on $\mathcal{X}^n\backslash\Delta$, $\mathcal{A}^n\backslash\Delta$ with Δ the fat diagonal. Make sure that you *check in detail* that:

- $\mathcal{L}(\prod_\nu M_\nu, N) \to \mathrm{Maps}(\prod_\nu \mathcal{V}_\nu, N)$, $f \mapsto f|_{\prod_\nu \mathcal{V}_\nu}$ is injective /// an isom if $\mathcal{V}_\nu = \mathcal{X}_\nu$ /// $\mathcal{V}_\nu = \mathcal{A}_\nu$
- $\mathcal{L}^n_{\text{sym}}(M;N) \to \mathrm{Maps}(\mathcal{V}^n_\leqslant, N)$, $f \mapsto f_{\mathcal{V}^n_\leqslant}$ is injective /// an isom if $\mathcal{V}^n_\leqslant = \mathcal{X}^n_\leqslant$ /// $\mathcal{V}^n_\leqslant = \mathcal{A}^n_\leqslant$
- $\mathcal{L}^n_{\text{alt}}(M;N) \to \mathrm{Maps}(\mathcal{V}^n_\leqslant, N)$, $f \mapsto f_{\mathcal{V}^n_\leqslant}$ is an injective /// an isom $\mathcal{V}^n_\leqslant = \mathcal{X}^n_\leqslant$ /// $\mathcal{V}^n_\leqslant = \mathcal{A}^n_\leqslant$

Finally recall that for $n = 2$ we speak about (symmetric/alternating) bilinear maps, and in the case $N = R, +$ we speak about bilinear/multilinear (symmetric/multilinear forms).

3) Let $n = 2$ and $|\mathcal{A}_\nu| = m_\nu$, $|\mathcal{A}| = m$ be finite, $N^{m_1\times m_2}$ be the $m_1 \times m_2$ matrices over N and define $\Psi : \mathcal{L}(M_1{\times}M_2, N) \to N^{m_1\times m_2}$, $f \mapsto A_f := \big(f(\alpha_{i_1}, \alpha_{i_2})\big)_{i_1,i_2}$. Prove/disprove:

a) $\Psi : \mathcal{L}(M_1{\times}M_2, N) \to N^{m_1\times m_2}$, $f \mapsto A_f$ is an isomorphism of R-modules.
Make an educated guess: *What happens with A_f under changes of bases $\mathcal{B}_\nu = S_\nu \mathcal{A}_\nu$?*

b) One has: $f \in \mathcal{L}^2_{\text{sym}}(M,N)$ iff A_f is symmetric; $f \in \mathcal{L}^2_{\text{alt}}(M,N)$ iff A_f is alternating.

c) Make an educated guess: *When do exist R-bases $\mathcal{A}$ s.t. $A_f := f(\alpha_i, \alpha_j)_{i,j}$ is diagonal?*

4) For $\mathcal{A}$ an R-basis of M, in the above notation, answer:

a) Given a map $f_0 : \mathcal{A}^n_{\leqslant} \to N$, define/describe $f \in \mathcal{L}^n_{\text{sym}}(M; N)$ satisfying $f|_{\mathcal{A}^n_{\leqslant}} = f_0$.

b) Given a map $f_0 : \mathcal{A}^n_{<} \to N$, define/describe $f \in \mathcal{L}^n_{\text{alt}}(M; N)$ satisfying $f|_{\mathcal{A}^n_{<}} = f_0$.

5) Let $\Sigma \subset R$ be a multiplicative system. Prove in all detail the assertions from class:

a) $R^+ \otimes_R M \cong_R M$ and $R_\Sigma \otimes_R M \cong_R M_\Sigma$ canonically (HOW).

b) $M \otimes_R N \cong_R N \otimes_R M$ canonically (HOW).

c) $(M \otimes_R N) \otimes_R P \cong_R M \otimes_R (N \otimes_R P)$ canonically (HOW).

d) $(M_1 \oplus M_2) \otimes_R N \cong_R (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$ (HOW). Hence $M \otimes_R N$ is R-free, if M, N are so. Make an educated guess: *Given R-bases $\mathcal{A}, \mathcal{B}$ of M, resp. N, give an R-basis of $M \otimes_R N$.*

6) Let S_1, S_2 be R-algebras, $S^+ = S_1^+ \otimes_R S_2^+$ endowed with $\imath_1 : S_1^+ \to S^+$, $s_1 \mapsto s_1 \otimes 1_{S_2}$, $\imath_2 : S_2^+ \to S^+$, $s_2 \mapsto 1_{S_1} \otimes s_2$. Prove in all detail the assertions from class:

a) The multiplication $(s_1 \otimes s_2) \cdot (s_1' \otimes s_2') := (s_1 s_1') \otimes (s_2 s_2')$ on $S = S_1^+ \otimes_R S_2^+$ is well defined, and $S = S_1 \otimes_R S_2$ becomes an R-algebra via $\varphi_S : R \to S$, $r \mapsto r(1_{S_1} \otimes 1_{S_2})$.

b) If S_1, S_2 are commutative, S endowed with $\imath_\nu : S_\nu \to S$, $\nu = 1, 2$ is the coproduct of S_1, S_2 in the category of commutative rings (allowing $R = \{0_R\}$ as object as well).

c) In general, $S = S_1 \otimes_R S_2$ endowed with $\imath_\nu$, $\nu = 1, 2$ is the the "commutative coproduct" of S_1, S_2 in the category of R-algebras (HOW).

7) Let S be a commutative R-algebra. Prove that the base change/extension of scalars R-**Mod** $\rightsquigarrow$ S-**Mod**, $M \mapsto M_S := S^+ \otimes_R M$ is a covariant functor. Further, prove/disprove:

a) If $M' \to M \to M''$ is exact in R-**Mod**, so is $M'_S \to M_S \to M''_S$ in S-**Mod**.

b) The same question for: (i) $S = R_\Sigma$ for $\Sigma \subset R$ a multiplicative system;
(ii) $S = R/\mathfrak{a}$ with $\mathfrak{a} \in \mathfrak{I}\text{d}(R)$ an ideal, e.g. $\mathfrak{a} = \mathcal{J}(R)$ the Jacobson radical of R.

c) Finally, the questions a), b) above for M', M, M'' finite R-modules.

8) Let M have an R-basis $\mathcal{A}$ with $|\mathcal{A}| = m$. Setting $\mathcal{L}^n_\bullet(M) := \mathcal{L}^n_\bullet(M; R)$, prove/disprove/answer:

a) $\mathcal{L}^n(M)$, $\mathcal{L}^n_{\text{sym}}(M)$, $\mathcal{L}^n_{\text{alt}}(M)$ have R-bases with m^n, $\binom{m+n-1}{n}$, $\binom{m}{n}$ entries, respectively.

b) *All R-bases $\mathcal{B}$ of M have $|\mathcal{B}| = m$.* **Terminology.** $|\mathcal{A}|$ is the rank $\text{rk}(M)$ of M.

c) Every system of generators $\mathcal{X}$ of M with $|\mathcal{X}| \leqslant m$ is an R-basis of M.

9) Determinant of $\varphi \in \text{End}_R(M)$, $A \in R^{m \times m}$. Given M in R-**Mod**, define an outer multiplication of $\text{End}_R(M), \circ$ on $\mathcal{L}^n_\bullet(M, N)$ by $\varphi \cdot f := f \circ \varphi^n$. Prove/disprove/answer:

a) The outer multiplication $f \cdot \varphi := f \circ \varphi^n$ is associative, i.e., $f \cdot (\psi \circ \varphi) = (f \cdot \psi) \cdot \varphi$.

b) Let M be R-free of rank $m = \text{rk}(M)$, hence $\mathcal{L}^m_{\text{alt}}(M) \cong_R R, +$ is R-free of rank 1 (WHY). Then for every $\varphi \in \text{End}_R(M)$ $\exists\, a_\varphi \in R$ unique s.t. $\varphi \cdot f = a_\varphi f$ for all $f \in \mathcal{L}^m_{\text{alt}}(M)$.

Terminology. The element $\det(\varphi) := a_\varphi \in R$ is called the determinant of $\varphi \in \text{End}_R(M)$.

c) $\det(\psi \circ \varphi) = \det(\psi) \det(\varphi) = \det(\varphi \circ \psi)$ and $\varphi \in \text{Aut}_R(M)$ iff $\det(\varphi) \in R^\times$.

d) If $A := (a_{ij})_{i,j} \in R^{m \times m}$ is the matrix of φ in some R-basis $\mathcal{A}$, i.e., $\varphi(\mathcal{A}) = \mathcal{A} \cdot A_\varphi$, then: $\det(A) := \det(\varphi) = \sum_{\sigma \in S_m} \text{sgn}(\sigma) a_{\sigma(1)1} \dots a_{\sigma(m)m} = \sum_{\tau \in S_m} \text{sgn}(\tau) a_{1\tau(1)} \dots a_{m\tau(m)} = \det(A^\tau)$.