Math 6030 / Problem Set 2 (two pages)

Pontrjagin Duality for abelian m-torsion groups. Let \( \mathcal{A} \) be the category of abelian torsion groups, considered as discrete topological groups, and \( \hat{\mathcal{A}} \) be the category of profinite abelian groups. Further, let \( \mathcal{A}_m \) and \( \hat{\mathcal{A}}_m \) be the corresponding full subcategories of \( m \)-torsion groups. Finally, let \( C_m \) be a typical cyclic group of order \( m \), e.g., \( C_m = \mathbb{Z}/m, \frac{1}{m}\mathbb{Z}/\mathbb{Z}, \mu_m \), which we consider as a discrete topological group.

For \( \Gamma \) in \( \mathcal{A}_m \) or in \( \hat{\mathcal{A}}_m \), let \( \mathcal{C}(\Gamma, C_m) \) be the space of continuous functions endowed with the open-compact topology. Not that \( \mathcal{C}(\Gamma, C_m) \) is a topological group (WHY). Finally, denote \( G^* := \text{Hom}(G, C_m) \subset \mathcal{C}(G, C_m) \) the group of continuous homomorphism.

1) Prove that \( G^* \subset \mathcal{C}(G, C_m) \) is topologically closed in \( \mathcal{C}(G, C_m) \), and further one has:
   a) If \( G \in \mathcal{A}_m \), then \( G^* \) is a profinite \( m \)-torsion group, hence \( G^* \in \hat{\mathcal{A}}_m \).
   Further, \( G \rightarrow G^* \) defines a contravariant functor \( \mathcal{A}_m \rightarrow \hat{\mathcal{A}}_m \).
   b) If \( G \in \hat{\mathcal{A}}_m \), then \( G^* \) is a discrete \( m \)-torsion group, hence \( G^* \in \mathcal{A}_m \).
   Further, \( G \rightarrow G^* \) defines a contravariant functor \( \hat{\mathcal{A}}_m \rightarrow \mathcal{A}_m \).
   c) Further, \( G \) and \( G^* \) are isomorphic as topological groups iff \( G \) id a finite.

2) Duality. Let \( \iota_G : G \rightarrow (G^*)^* = \text{Hom}(G^*, C_m), g \mapsto \Phi_g, \Phi_g(\varphi) = \varphi(g) \forall \varphi \in G^* \). Prove:
   a) The canonical map \( \iota_G : G \rightarrow (G^*)^* \) is an isomorphism of topological groups, hence:
      \( \hat{\mathcal{A}}_m \rightarrow \mathcal{A}_m \) and \( \mathcal{A}_m \rightarrow \hat{\mathcal{A}}_m \) are inverse to each other, thus \( \mathcal{A}_m, \hat{\mathcal{A}}_m \) are (anti)equivalent.
   b) The subgroups of \( G \) correspond functorially to the factor groups of \( G^* \).

Artin–Schreier Thm and (formally) real fields. Recall that a (formally) real field is any field \( K \) which admits a total ordering \( \leq \) which is compatible with the field operations (HOW). Obviously, in a real field \( K \) one has \(-1_K < 0_K < 1_K \) (WHY), hence \( \text{char}(K) = 0 \) (WHY). A real field is called real closed if there is no proper algebraic extension \( K' | K \) of real fields, i.e., \( K' \) carrying a total field ordering \( \leq' \) which prolongs \( \leq \) to \( K' \). It turns out the total orderings of fields relate in a subtle with the (sums of) squares in \( K \). In the sequel, \( K^{\ast,2} := \{ x^2 \mid x \in K \} \) is the set of squares and \( \sum K^{\ast,2} := \{ \sum_i x_i^2 \mid x_i \in K \} \) is the set of finite sums of squares in \( K \).

3) Without invoking the Artin–Schreier Thm, setting \( i = \sqrt{-1} \), prove directly:
   a) TFAE: (i) \( 1 < [K : K] < \infty \); (ii) \( 1 < [K^s : K] < \infty \); (iii) \( K \neq K^s = K(i), \text{char}(K) = 0 \).
   b) If (i), (ii), (iii) are satisfied, then \( K^{\times} = -K^{\ast,2} \cup K^{\ast,2}, -K^{\ast,2} \cap K^{\ast,2} = \{ 0 \}, \sum K^{\ast,2} = K^{\ast,2} \).
Conclude: \( x \leq y \overset{\text{def}}{\iff} y - x \in K^{\ast,2} \) defines a total field ordering on \( K \).

4) Prove that for an arbitrary field \( K \) one has: \( \sum K^{\ast,2} \) is a semifield, i.e., it is closed w.r.t.
   +, \cdot and inverses \( x^{-1} \) for nonzero \( x \in \sum K^{\ast,2} \). Invoking this fact, prove:

Artin’s Theorem. \( K \) is a real field if and only if \(-1 \not\in \sum K^{\ast,2} \).

[HInt to Artin’s Thm: First, if \( K \) is real, then \(-1 \not\in \sum K^{\ast,2} \) (WHY) and \( \text{char}(K) = 0 \) (WHY). For the converse prove:
   - Let \( K_1 | K \) be an algebraic extension with \([K_1 : K] \) odd. Then \( \sum K_1^{\ast,2} \subset K_1^{\times} \) is a semifield and \(-1 \not\in \sum K^{\ast,2} \).
   - Let \( K_2 = K[\sqrt{\sum K^{\ast,2}}] \). Then \( \sum K_2^{\ast,2} \subset K_2^{\ast} \) is a semifield and \(-1 \not\in \sum K^{\ast,2} \).
Conclude: If \( \bar{K} \subset K \) is a maximal real subfield, then \( \bar{K} = \bar{K}[\sqrt{-1}], \) hence \( K \) is a real closed by Problem 3) above.]
Basics about $\text{Max}_R(R), \text{Spec}(R) \subset \mathfrak{Jd}(R)$.
Recall that for a ring $R$ we defined $\text{Max}_R(R) \subset \mathfrak{Jd}_R(R), \text{Spec}(R)$, where • can be $l$ (left), $r$ (right), or empty, and the latter case, $\text{Max}(R) \subset \mathfrak{Jd}(D)$, means two-sided. Further, $\mathcal{J}(R) = \cap_{m \in \text{Max}_R(R)} m$ is the Jacobson radical of $R$, and $\mathcal{N}(R) = \cap_{p \in \text{Spec}(R)} p$ is the nil-radical of $R$. Similarly, given $a \in \mathfrak{Jd}_R(R)$, $\mathcal{J}(a) = \cap_{m \in \text{Max}_R(R)} m$ is the Jacobson radical of $a$, and $\mathcal{N}(a) = \cap_{p \in \text{Spec}(R)} p$ is the nil-radical of $a$.

5) For an arbitrary ring $R$ with $1_R$ prove/disprove/answer:
   a) $\text{Max}(R) \subset \text{Spec}(R)$ and $\mathcal{J}(m) = m$ for all $m \in \text{Max}(R)$.
   b) Every $m \in \text{Max}_R(R)$ contains maximal ideals $p \subset m$, and all such $p$ are prime ideals.
   c) How do $\mathcal{N}(R)$ and $\mathcal{J}(R)$ compare?

6) For an arbitrary ring $R$ with $1_R$ prove/disprove/answer:
   a) Let $a_1, \ldots, a_n \in \mathfrak{Jd}(R)$, $p \in \text{Spec}(R)$. If $\cap_i a_i \subset p$, then $\exists i_0$ such that $a_{i_0} \subset p$.
      The same question with “$\subset$” replaced by “$=$”.
   b) Let $p_1, \ldots, p_n \in \text{Spec}(R), a \in \mathfrak{Jd}(R)$. If $a \subset \cup_i p_i$, then $\exists i_0$ such that $a \subset p_{i_0}$.
      What is the corresponding assertion with “$\subset$” replaced by “$=$” ?

7) For an arbitrary ring $R$ with $1_R$ and $a, a_i \in \mathfrak{Jd}(R)$, prove/disprove/answer:
   a) $\mathcal{N}(\mathcal{N}(a)) = \mathcal{N}(a)$, $\mathcal{N}(\sum_i a_i) = \mathcal{N}(\sum_i \mathcal{N}(a_i))$.
   b) $\mathcal{N}(\prod_i a_i) = \cap_i \mathcal{N}(a_i) = \mathcal{N}(\cap_i a_i)$.
   (*) What are the corresponding assertions for the Jacobson radical $\mathcal{J}(\bullet)$?

Extension/Contraction of Ideals.
Let $f : R \to S$ be a ring morphism with $f(1_R) = 1_S$, and recall the •-ideal extension map $f_* : \mathfrak{Jd}_*(R) \to \mathfrak{Jd}_*(S)$, $a \mapsto a^e := (f(a))^e_*$, respectively the •-ideal contraction map $f^* : \mathfrak{Jd}_*(S) \to \mathfrak{Jd}_*(R), b \mapsto b^c := f^{-1}(b)$. These maps are well defined. Further, let $\mathfrak{Jd}_e(R) \subset \mathfrak{Jd}_*(R)$ be the subset of ideals which are contracted, and $\mathfrak{Jd}_e(S) \subset \mathfrak{Jd}_*(S)$ be the subset of ideals which are extended.

8) For $a, a_i \in \mathfrak{Jd}_e(R)$ and $b, b_i \in \mathfrak{Jd}_e(S)$, prove/disprove the assertions (same made in class):
   a) $a^{ec} := (a^e)^c \supset a$ and $b^{ec} := (b^c)^e \subset b$.
   b) $f_* : \mathfrak{Jd}_e(R) \to \mathfrak{Jd}_e(S)$ and $f^* : \mathfrak{Jd}_e(S) \to \mathfrak{Jd}_e(R)$ are well defined bijections and $f^* = f_*^{-1}$.
   c) $\left( \sum_i a_i \right)^c = \sum_i a_i^c$ and $\left( \prod_i a_i \right)^c = \prod_i a_i^c$. Further, $\left( \sum_i b_i \right)^c = \sum_i b_i^c$ and $\left( \prod_i b_i \right)^c = \prod_i b_i^c$.
      The same questions for $a_i \in \mathfrak{Jd}_e(R)$, respectively $b_i \in \mathfrak{Jd}_e(S)$.

9) In the above notation, prove/disprove/answer the following:
   a) If $q \in \text{Spec}(S)$, then $q^e \in \text{Spec}(R)$, i.e., contractions of prime ideals are prime ideals.
      Hence $f^* : \text{Spec}(S) \to \text{Spec}(R), q \mapsto p := q^e = f^{-1}(q)$ is well defined.
   b) For $b \in \mathfrak{Jd}(S)$ one has $\mathcal{N}(b)^c = \mathcal{N}(b^c)$. The same question for $b \in \mathfrak{Jd}^e(S)$.

10) Give examples to show that, in general, maximal ideals do not behave well under extension and/or contraction, and that prime ideals do not behave well under extension.