Due: April 26, 2024

Math 6030 / Problem Set 11 (two pages)

Miscellaneous

Recall: for a commutative ring R with 1_R and $\mathfrak{a} \in \mathcal{I}d(R)$, we denote by $\mathcal{N}(\mathfrak{a}) \subset \mathcal{J}(\mathfrak{a})$ the nil radical, resp. Jacobson radical of $\mathfrak{a}$. Equivalently, if $pr : R \to \overline{R} = R/\mathfrak{a}$ and $\mathcal{N}(\overline{R}) \subset \mathcal{J}(\overline{R})$ are the nil, resp. Jacobson radical of $\overline{R}$, then $\mathcal{N}(\mathfrak{a}) = pr^{-1}\big(\mathcal{N}(\overline{R})\big)$ and $\mathcal{J}(\mathfrak{a}) = pr^{-1}\big(\mathcal{J}(\overline{R})\big)$.

1) Prove that for a commutative ring R with 1_R, TFAE:

(i) For every $\mathfrak{p} \in \operatorname{Spec}(R)$ one has $\mathfrak{p} = \mathcal{J}(\mathfrak{p})$.
(ii) For every $\mathfrak{a} \in \mathcal{I}d(R)$ one has $\mathcal{N}(\mathfrak{a}) = \mathcal{J}(\mathfrak{a})$.
(iii) For every surjective ring morphism $R \twoheadrightarrow S$ one has $\mathcal{N}(S) = \mathcal{J}(S)$.

Terminology. R satisfying the equivalent conditions (i), (ii), (iii) above is a Jacobson ring.

2) Let I be a nonempty (finite or infinite) set. Prove/disprove/answer:

a) A polynomial ring $k[t_i]_{i\in I}$ is a Jacobson ring, provided: (i) I is finite; (ii) I is arbitrary.
b) Same questions for the k-algebras $R = k[x_i]_{i\in I}$ in I-generators.
c) Same questions for the polynomial ring $\mathbb{Z}[t_i]_{i\in I}$, respectively the $\mathbb{Z}$-algebra $\mathbb{Z}[x_i]_{i\in I}$.

Let k be a base field, $\underline{\boldsymbol{X}} = (X_1, \ldots, X_n)$ be k-independent variables, and recall that every ideal $\mathfrak{a} \subset k[\underline{\boldsymbol{X}}]$ is finitely generated (WHY), hence of the form $\mathfrak{a} = (f_1,\ldots, f_r)$ for some $f_j \in k[\underline{\boldsymbol{X}}]$. Given $\mathfrak{a} \in \mathcal{I}d\big(k[\underline{\boldsymbol{X}}]\big)$, set $R := k[\underline{\boldsymbol{X}}]/\mathfrak{a} = k[x_1,\ldots, x_n]$ with $x_i := X_i (\operatorname{mod} \mathfrak{a})$. Let $K|k$ a field extension with $K = \overline{K}$ algebraically closed, and $\overline{k}|k \hookrightarrow K|k$ the algebraic closure of k in K. Given a set $\boldsymbol{f} = \{f_i\}_i$ of polynomials $f_i \in k[\underline{\boldsymbol{X}}]$, denote $V(\boldsymbol{f}) := \{\boldsymbol{a} \in K^n \mid f_i(\boldsymbol{a}) = 0 \,\forall\, i\}$ and call $V = V(\boldsymbol{f}) \subset K^n$ an k-algebraic (sub)set of K^n. Finally, given a k-algebraic subset $V \subset K^n$, we denote $I(V) := \{f \in k[\underline{\boldsymbol{X}}] \mid f(\boldsymbol{a}) = 0 \,\forall\, \boldsymbol{a} \in V\}$ and call it ideal of V.

3) Let $V, W \subset K^n$ be k-algebraic subsets. Prove/disprove/answer:

a) Given $\boldsymbol{f}$, let $\mathfrak{a}_{\boldsymbol{f}} \subset k[\underline{\boldsymbol{X}}]$ be the ideal generated by $\boldsymbol{f}$. Then $V(\boldsymbol{f}) = V(\mathfrak{a}_{\boldsymbol{f}})$.
b) $I(V) \subset k[\underline{\boldsymbol{X}}]$ is an ideal, and further one has $I(V) = \mathcal{N}(\mathfrak{a}_{\boldsymbol{f}}) = \mathcal{J}(\mathfrak{a}_{\boldsymbol{f}})$
c) $V = W$ iff $I(V) = I(W)$ iff $V \cap \overline{k}^n = W \cap \overline{k}^n$.

4) Let $\operatorname{td}(K|k)$ be the transcendence degree of $K|k$. Prove/disprove/answer:

a) The map $\operatorname{Hom}_k\big(k[\underline{\boldsymbol{X}}], K\big) \to K^n,\ \varphi \mapsto \boldsymbol{a} := \big(\varphi(X_1), \ldots, \varphi(X_n)\big)$ is a bijection.
b) For every $\varphi \in \operatorname{Hom}_k\big([\underline{\boldsymbol{X}}], K\big)$ one has $\mathfrak{p}_\varphi := \ker(\varphi) \in \operatorname{Spec}\big(k[\underline{\boldsymbol{X}}]\big)$.
c) For $\mathfrak{p} \in \operatorname{Spec}\big(k[\underline{\boldsymbol{X}}]\big)$ $\exists\, \varphi \in \operatorname{Hom}_k(k[\underline{\boldsymbol{X}}], K)$ with $\mathfrak{p} = \mathfrak{p}_\varphi$ iff $\operatorname{coht}(\mathfrak{p}) \leqslant \operatorname{td}(K|k)$.

Integral ring extensions/Hilbert Decomposition Theory

Recall the basics: Let G be a profinite group, N_i, $i \in I$ denote its open normal subgroups, $pr_i : G \twoheadrightarrow G_i = G/N_i$, $g \mapsto g_i$ the canonical projections, hence $G = \varprojlim_i G_i$ canonically (HOW). Let S be a discrete ring on which G acts continuously. Equivalently, the orbits Gx, $x \in S$ are finite (WHY). For every N_i, set $S_i := S^{N_i} := \{x \in S \mid N_i x = x\}$, and $R := S^G$. Then $S_j \supset S_i$ iff $N_j \subset N_i$ (WHY), and consider the restriction maps $i^*_{ji} : \mathcal{I}d(S_j) \to \mathcal{I}d(S_i)$, $\mathfrak{a}_j \mapsto \mathfrak{a}_i := \mathfrak{a}_j \cap S_i$.

Recall that $S|R$ is integral (WHY), and so are $S|S_i$, $S_j|S_i$ and $S_i|R$ (WHY), and recall the maps $X_\mathfrak{p} \twoheadrightarrow X^j_\mathfrak{p} \twoheadrightarrow X^i_\mathfrak{p}$ of sets of primes $X_\mathfrak{p} \subset \mathrm{Spec}(S)$, $X^\bullet_\mathfrak{p} \subset \mathrm{Spec}(S_\bullet)$, above a given $\mathfrak{p} \in \mathrm{Spec}(R)$.

5) In the above notation/context, prove/disprove/answer:

a) G_i acts on S_i by $g_i x := gx$, where $g \mapsto g_i$ under $pr_i : G \twoheadrightarrow G_i$, and $S_i^{G_i} = R$.

b) $S = \cup_i S_i$, hence if $\mathfrak{a} \in \mathcal{I}d(S)$ and $\mathfrak{a}_i := \mathfrak{a} \cap S_i$, then $\mathfrak{a} = \cup_i \mathfrak{a}_i$.

c) $\big(\mathcal{I}d(S_i), \imath^*_{kj}\big)_{i,k\geqslant j}$ is a projective system of sets, and $\mathcal{I}d(S) = \varprojlim_i \mathcal{I}d(S_i)$ (HOW).

d) $\imath^*_{ji}$ is compatible with the groups action, i.e., if $g_j \mapsto g_i$, then $\imath^*_{ji}\big(g_j(\mathfrak{a}_j)\big) = g_i(\mathfrak{a}_i)$.

6) Recalling that $\mathrm{Spec}(\bullet)$ carries the Zariski topology, prove/disprove:

a) $\mathrm{Spec}(S) \to \mathrm{Spec}(S_i) \to \mathrm{Spec}(R)$ are onto, continuous, compatible with group actions.

b) $\big(X^i_\mathfrak{p}, \imath^*_{kj}\big)_{i,k\geqslant j}$ is a projective surjective system of finite sets, and $X_\mathfrak{p} = \varprojlim_i X^i_\mathfrak{p}$.

Conclude: $X_\mathfrak{p} \subset \mathrm{Spec}(S)$ is a *profinite topological space*—as a subspace of $\mathrm{Spec}(S)$.

7) In the above context, for $\mathfrak{q} \mapsto \mathfrak{q}_i \mapsto \mathfrak{p}$, prove/disprove:

a) $D_{\mathfrak{q}|\mathfrak{p}} \twoheadrightarrow D_{\mathfrak{q}_i|\mathfrak{p}}$ under $G \twoheadrightarrow G_i$, and $D_\mathfrak{q} = \varprojlim_i D_{\mathfrak{q}_i|\mathfrak{p}}$.

b) G acts continuously on the profinite space $X_\mathfrak{p}$, and $X_\mathfrak{p} \cong G/D_{\mathfrak{q}|\mathfrak{p}}$ as G-spaces.

8) In the above notation and context, $(\mathfrak{q}_j)_j$ and $(\mathfrak{p}_i)_i$ be maximal chains in $\mathrm{Spec}(S)$, respectively in $\mathrm{Spec}(R)$. Prove/disprove:

a) Setting $\mathfrak{p}_j := \mathfrak{q}_j \cap R$, the chain $(\mathfrak{p}_j)_j$ is maximal in $\mathrm{Spec}(R)$.

b) There is a maximal chain $(\mathfrak{q}_i)_i$ in $\mathrm{Spec}(S)$ with $\mathfrak{p}_i = \mathfrak{q}_i \cap R$.

$*$) Same questions for any subring $S' \subset S$ which contains R, i.e., $R \subset S' \subset S$.

Fractional Ideals

Let R be a commutative ring R with 1_R and $K = R_{\Sigma^0_R}$ be its total ring of fractions. An R-submodule $M \subset K,+$ is a fractional ideal (of R) if $\exists\, r \in \Sigma^0_R$ such that $rM \subset R$. Recall that given fractional ideals M, N of R, one defines $(M : N) := \{x \in K \,|\, xN \subset M\}$ and $M \cdot N = \langle xy \,|\, x \in M, y \in N \rangle_R \subset K,+$ the corresponding R-submodules. Finally, a fractional ideal M is called invertible, if there is a fractional R-submodule $N \subset K,+$ such that $M \cdot N = R$.

Notation. Let $\mathcal{I}'_R$ be the set of fractional ideals, $\mathcal{I}_R \subset \mathcal{I}'_R$ be the invertible fractional ideals.

9) In the above notation, Prove/disprove/answer:

a) If $M, N \in I'_R$, then $M + N$, $M \cdot N$, $(M : N)$ are fractional ideals.

b) I'_R endowed with $+$ and $\cdot$ is a semi-ring.

c) $I_R \subset I'_R$ is the group of invertible elements in the monoid $I'_R, \cdot$ of fractional ideals.

[**Hint** to a): Use that $xR \in \mathcal{I}_R$ for all $x \in K^\times$ (WHY) and $N' \subset N$, $M' \subset M \Rightarrow (M' : N) \subset (M : N)$, $(M : N) \subset (M : N')$ (WHY), etc.]

10) Let M be a fractional ideal, prove/disprove:

a) If $M \in \mathcal{I}_R$, then M is a finite R-module, and $\exists\, s \in M \cap \Sigma^0_R$.

b) M is invertible iff $M \cdot (R : M) = R$. Hence $M \in \mathcal{I}_R$ iff $(R : M)$ is the inverse of M.

c) TFAE: (i) $M \in I_R$; (ii) $M_\mathfrak{p} \in I_{R_\mathfrak{p}}\ \forall\, \mathfrak{p} \in \mathrm{Spec}(R)$; (iii) $M_\mathfrak{m} \in I_{R_\mathfrak{m}}\ \forall\, \mathfrak{m} \in \mathrm{Max}(R)$.