Due: Friday, May 10, 2024, at 12(noon)

Math 6030 / Final Exam (two pages)

Academic Integrity Statement:

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this Math 6030 exam. [That means, among other things, that you are allowed to: (a) get hints from your colleagues, but do not work out solutions together; (b) ask any member the Math Dept about hints to the exam problems, but you must *first* mention to her/him that it goes about problems on a take home exam; (c) consult resources (books, Internet, etc.), but you must design your own proofs, and not copy word-by-word from other sources.]

Name (printed): ______________________ Signature: ____________________ Date ________

Note: There are 9(nine) problems on this exam.

Points: Separate for each problem (extra and/or partial credit possible).

Grading: D $<$ 60 $\leqslant$ C−, C, C+ $<$ 75 $\leqslant$ B−, B, B+ $<$ 90 $\leqslant$ A−, A, A+

Procedures: Write your name (printed) and sign the above Academic Integrity Statement.
Indicate clearly the number of each problem you work out.
Your submission should show the necessary work/ideas, but be concise.

• Recall: A complete proof must contain all the necessary explanations/steps, and in order to *disprove* an assertion you must give a counterexample showing that the assertion is not true.

1) (12 pts) Prove/disprove/answer the following (justify your answer!):

a) Enumerate up to isomorphism the semisimple rings R with 1_R of cardinalities $6, 9, 12, 16$.

b) Find a Galois extension $K|\mathbb{Q}$ of degree 13 s.t. the trace $\mathrm{Tr}_{K|\mathbb{Q}}\colon K \to \mathbb{Q}$ is not surjective.

c) Give a non-maximal prime ideal $\mathfrak{p} \subset \mathbb{Z}[t]$ which is not principal.

2) (12 pts) Let k be a field and $S = k[x_1, x_2] = k[t_1, t_2]/\mathfrak{a}$, where $\mathfrak{a} = (t_1^2 t_2 - t_1 + t_2)$ and $x_i := t_i (\mathrm{mod}\, \mathfrak{a})$, $i = 1, 2$. Setting $R := k[t]$, $S_1 = k[x_1]$, prove/disprove/answer the following:

a) S is an integral domain.

b) $R \to S_1$, $t \mapsto x_1$ is an isomorphism, and $S \subset \mathrm{Quot}(S_1) = k(x_1)$.

c) Describe the integral closure $\widetilde{S}$ of S in $\mathrm{Quot}(S_1) = k(x_1)$.

[**Hint** to b): $x_1^2 x_2 - x_1 + x_2 = 0 \Rightarrow x_2 = \frac{x_1}{x_1^2+1} \in S \subset k(x_1)$ (WHY), etc. To c): Show that $S = [x_1, \frac{1}{x_1^2+1}]$, hence integrally closed (WHY), etc. ...]

3) (10 pts) In the context from Problem 2) above, define $\varphi : R \to S$, $t \mapsto x_2$. Prove/disprove/answer:

a) First, φ is injective. Second, $S|R$ is an integral ring extension under φ.

c) For $k = \overline{k}$, compute the fibers of $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ and of $\imath^* : \mathrm{Spec}(\widetilde{S}) \to \mathrm{Spec}(S)$.

4) (14 pts) Let $R \subset \mathbb{Q}$ be a subring strictly containing $\mathbb{Z}$, S be a commutative ring with $1_S \neq 0_R$, and N be a free S-module. Prove/disprove:

a) $Q := (R,+)/(\mathbb{Z},+)$ is an injective module in the category of $\mathbb{Z}$-modules.

b) $N_Q := \mathrm{Hom}_{\mathbb{Z}}(N,Q)$ is an injective S-module under $r\cdot\varphi(x) := \varphi(rx)\,\forall\, r\in S, x\in N$.

[**Hint**: $M^D := \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ is an injective S-module for every free S-module M (WHY). Further, every $R\subset\mathbb{Q}$ is a fraction ring $R=\mathbb{Z}_\Sigma$ with $\Sigma\subset\mathbb{N},\cdot$ generated by primes, and $\Sigma\cup\Sigma' = \mathfrak{Primes} \Rightarrow \mathbb{Z}_\Sigma + \mathbb{Z}_{\Sigma'} = \mathbb{Q}$, $\mathbb{Z}_\Sigma\cap\mathbb{Z}_{\Sigma'} = \mathbb{Z}_{\Sigma\cap\Sigma'}$ (WHY), etc. ...]

5) (12 pts) Which of the following is a PID/UFD/Noetherian/Artinian/valuation ring?

a) $R = F[t_1,\dots,t_d]_{\mathfrak{p}}$, where F is a field and $\mathfrak{p} = (t_1,\dots,t_r)$ for some $r\leqslant d$.

b) $R = \mathbb{Z}[t_1,t_2]/(t_1^2t_2^3 - 3, t_1^2 - t_2^4)$.

[**Hint** to b): R is finitely generated over $\mathbb{Z}$ (WHY), not a domain (WHY), and $\mathrm{Krulldim}(R) > 0$ (WHY), etc. ...]

6) (12 pts) Let R be a commutative ring with 1_R, $R[[t]]$ be the power series ring, $S = R[x_1,\dots,x_n]$ be a finitely generated commutative R-algebra. Prove/disprove:

a) The nil radical $\mathcal{N}(R[[t]])$ is nilpotent iff the nil radical $\mathcal{N}(R)$ is nilpotent.

b) If S is Noetherian then R is Noetherian. Same question provided $S|R$ is ring extension.

[**Hint** to b): There are "many" domains R and $x\in R$ s.t. $S = R[x_1] = \mathrm{Quot}(R)$, where $x_1 = \frac{1}{x}$ (WHY)...]

7) (14 pts) Let R be a Dedekind ring, $K = \mathrm{Quot}(R)$. Prove/disprove:

a) There is a proper integral ring extension $S|R$ with $S\subset K(t)$.

b) One has $\mathrm{Max}(R[t])\cap R \neq \mathrm{Max}(R)$.

8) (10 pts) For a field k, let $R = k[t_1,t_2,t_3]$, $\mathfrak{a} = (t_4^3 - t_1t_2t_3)\subset R[t_4]$ and consider the factor ring $S = R[t_4]/(t_4^3 - t_1t_2t_3) = k[x_1,x_2,x_3,x_4]$, where $x_i = t_i (\mathrm{mod}\ \mathfrak{a})$. Prove/disprove/answer:

a) If $k=\mathbb{R}$, then every $\mathfrak{n}\in\mathrm{Max}(S)$ is of the form $\mathfrak{n} = (x_i - a_i)_{1\leqslant i\leqslant 4}$ with $a_i\in\mathbb{R}$.

b) There are fields k for which some prime ideals $\mathfrak{p}\in\mathrm{Spec}(S)$ have height $\mathrm{ht}(\mathfrak{p}) = 4$.

[**Hint** to a): For $\boldsymbol{a} = (a_1,a_2,a_3,a_4)\in\overline{k}$ let $\varphi_{\boldsymbol{a}} : k[t_1,t_2,4_3,t_4]\to\overline{k}$, $(t_1,t_2,4_3,t_4)\mapsto\boldsymbol{a}$. Then $\mathfrak{m}_{\boldsymbol{a}} = \mathrm{Ker}(\varphi_{\boldsymbol{a}})\in\mathrm{Max}(k[t_1,t_2,t_3,t_4])$ (WHY), and $\mathfrak{m}_{\boldsymbol{a}}$ defines a (maximal) ideal of S iff $a_4^3 - a_1a_2a_3 = 0$ (WHY), etc... To b): $\mathrm{ht}(\mathfrak{p})\leqslant\mathrm{Kulldim}(S)$ (WHY), etc...]

9) (8 pts) In the context of Problem 8) above, let $k=\overline{k}$, and $K := \mathrm{Quot}(R)$, $L := \mathrm{Quot}(S)$.

a) $L|K$ is Galois and $G(L|K)$ acts on S, i.e. $\sigma(S) = S$ for all $\sigma\in G(L|K)$.

b) Given $\mathfrak{m}\in\mathrm{Max}(R)$, describe the possible decomposition groups $D_{\mathfrak{n}|\mathfrak{m}}$ for $\mathfrak{n}\in X_{\mathfrak{m}}$.

[**Hint** to b): How many solutions has the equation $a_4^3 - a_1a_2a_3 = 0$ for $a_1,a_2,a_3\in k$ arbitrary?]