Due: Friday, May 10, 2024, at 12(noon)

Math 6030 / Final Exam (two pages)

Academic Integrity Statement:

My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this Math 6030 exam. [That means, among other things, that you are allowed to: (a) get hints from your colleagues, but do not work out solutions together; (b) ask any member the Math Dept about hints to the exam problems, but you must first mention to her/him that it goes about problems on a take home exam; (c) consult resources (books, Internet, etc.), but you must design your own proofs, and not copy word-by-word from other sources.]

Name (printed): ______________________ Signature: ______________________ Date ______

Note: There are 9(nine) problems on this exam.

Points: Separate for each problem (extra and/or partial credit possible).

Grading: 
D < 60 ≤ C−, C, C+ < 75 ≤ B−, B, B+ < 90 ≤ A−, A, A+

Procedures: Write your name (printed) and sign the above Academic Integrity Statement.
Indicate clearly the number of each problem you work out.
Your submission should show the necessary work/ideas, but be concise.

• Recall: A complete proof must contain all the necessary explanations/steps, and in order to disprove an assertion you must give a counterexample showing that the assertion is not true.

1) (12pts) Prove/disprove/answer the following (justify your answer!):
   a) Enumerate up to isomorphism the semisimple rings $R$ with $1_R$ of cardinalities 6, 9, 12, 16.
   b) Find a Galois extension $K|\mathbb{Q}$ of degree 13 s.t. the trace $\text{Tr}_{K|\mathbb{Q}}: K \to \mathbb{Q}$ is not surjective.
   c) Give a non-maximal prime ideal $p \subset \mathbb{Z}[t]$ which is not principal.

2) (12pts) Let $k$ be a field and $S = k[x_1, x_2] = k[t_1, t_2]/a$, where $a = (t_1^2 t_2 - t_1 + t_2)$ and $x_i := t_i \pmod{a}, i = 1, 2$. Setting $R := k[t], S_1 = k[x_1]$, prove/disprove/answer the following:
   a) $S$ is an integral domain.
   b) $R \to S_1, t \mapsto x_1$ is an isomorphism, and $S \subset \text{Quot}(S_1) = k(x_1)$.
   c) Describe the integral closure $\tilde{S}$ of $S$ in $\text{Quot}(S_1) = k(x_1)$.
   [Hint to b): $x_1^2 x_2 - x_1 + x_2 = 0 \Rightarrow x_2 = \frac{x_1}{x_1^2 + 1} \in S \subset k(x_1)$ (WHY), etc. To c): Show that $S = [x_1, \frac{1}{x_1^2 + 1}],$ hence integrally closed (WHY, etc. . . . )]

3) (10pts) In the context from Problem 2) above, define $\varphi : R \to S, t \mapsto x_2$. Prove/disprove/answer:
   a) First, $\varphi$ is injective. Second, $S|R$ is an integral ring extension under $\varphi$.
   c) For $k = \overline{k}$, compute the fibers of $\varphi^* : \text{Spec}(S) \to \text{Spec}(R)$ and of $\nu^* : \text{Spec}(\tilde{S}) \to \text{Spec}(S)$.

4) (14pts) Let $R \subset \mathbb{Q}$ be a subring strictly containing $\mathbb{Z}$, $S$ be a commutative ring with $1_S \neq 0_R$, and $N$ be a free $S$-module. Prove/disprove:
a) \( Q := (R, +)/(\mathbb{Z}, +) \) is an injective module in the category of \( \mathbb{Z} \)-modules.

b) \( N_Q := \text{Hom}_{\mathbb{Z}}(N, Q) \) is an injective \( S \)-module under \( r \cdot \varphi(x) := \varphi(rx) \forall r \in S, x \in N. \)

[Hint: \( M^D := \text{Hom}_2(M, Q/\mathbb{Z}) \) is an injective \( S \)-module for every free \( S \)-module \( M \). Further, every \( R \subset Q \) is a fraction ring \( R = \mathbb{Z}_S \) with \( \Sigma \subset \mathbb{N} \cdot \) generated by primes, and \( \Sigma \cup \Sigma' = \text{primes} \Rightarrow \mathbb{Z}_S + \mathbb{Z}_{S'} = Q, \mathbb{Z}_S \cap \mathbb{Z}_{S'} = \mathbb{Z}_{S \cap S'} \) (WHY), etc. . . .]

5) (12pts) Which of the following is a PID/UFD/Noetherian/Artinian/valuation ring?

a) \( R = F[t_1, \ldots, t_d]_p \), where \( F \) is a field and \( p = (t_1, \ldots, t_r) \) for some \( r \leq d. \)

b) \( R = \mathbb{Z}[t_1, t_2]/(t_1^2 + 3, t_1^2 - t_2^2). \)

[Hint to b): \( R \) is finitely generated over \( \mathbb{Z} \) (WHY), not a domain (WHY), and \( \text{Krulldim}(R) > 0 \) (WHY), etc. . . .]

6) (12pts) Let \( R \) be a commutative ring with \( 1_R \), \( R[[t]] \) be the power series ring, \( S = R[x_1, \ldots, x_n] \) be a finitely generated commutative \( R \)-algebra. Prove/disprove:

a) The nil radical \( \mathcal{N}(R[[t]]) \) is nilpotent iff the nil radical \( \mathcal{N}(R) \) is nilpotent.

b) If \( S \) is Noetherian then \( R \) is Noetherian. Same question provided \( S \mid R \) is ring extension.

[Hint to b): There are “many” domains \( R \) and \( x \in R \) s.t. \( S = R[x_1] = \text{Quot}(R) \), where \( x_1 = \frac{1}{x} \) (WHY) . . .]

7) (14pts) Let \( R \) be a Dedekind ring, \( K = \text{Quot}(R) \). Prove/disprove:

a) There is a proper integral ring extension \( S \mid R \) with \( S \subset K(t) \).

b) One has \( \text{Max}(R[[t]]) \cap R \neq \text{Max}(R) \).

8) (10pts) For a field \( k \), let \( R = k[t_1, t_2, t_3] \), \( a = (t_1^3 - t_1^2 t_2 t_3) \subset R[t_4] \) and consider the factor ring \( S = R[t_4]/(t_4^3 - t_1^2 t_2 t_3) = k[x_1, x_2, x_3, x_4] \), where \( x_i = t_i \pmod{a} \). Prove/disprove/answer:

a) If \( k = \mathbb{R} \), then every \( n \in \text{Max}(S) \) is of the form \( n = (x_i - a_i)_{1 \leq i \leq 4} \) with \( a_i \in \mathbb{R} \).

b) There are fields \( k \) for which some prime ideals \( p \in \text{Spec}(S) \) have height \( \text{ht}(p) = 4. \)

[Hint to a): For \( a = (a_1, a_2, a_3, a_4) \in \mathbb{K} \) let \( \varphi_a : k[t_1, t_2, 4, t_4] \rightarrow \mathbb{K}, (t_1, t_2, 4, t_4) \rightarrow a \). Then \( m_a = \text{Ker}(\varphi_a) \in \text{Max}(k[t_1, t_2, 4, t_4]) \) (WHY), and \( m_a \) defines a (maximal) ideal of \( S \) iff \( a_1^3 - a_1 a_2 a_3 = 0 \) (WHY), etc. . . . To b): \( \text{ht}(p) \leq \text{Kulldim}(S) \) (WHY), etc. . . .]

9) (8pts) In the context of Problem 8) above, let \( k = \mathbb{K} \), and \( K := \text{Quot}(R), L := \text{Quot}(S) \).

a) \( L \mid K \) is Galois and \( G(L \mid K) \) acts on \( S \), i.e. \( \sigma(S) = S \) for all \( \sigma \in G(L \mid K) \).

b) Given \( m \in \text{Max}(R) \), describe the possible decomposition groups \( D_{nm} \) for \( n \in X_m \).

[Hint to b): How many solutions has the equation \( a_1^3 - a_1 a_2 a_3 = 0 \) for \( a_1, a_2, a_3 \in k \) arbitrary? ]