Math 6020 / Problem Set 9 (two pages)

Group extensions

Recall: For a group $G$ acting on a group $A$ via a fixed group morphism $\varphi : G \to \operatorname{Aut}(A)$, $\sigma \mapsto \varphi_\sigma$, denote $\varphi_\sigma(a) =: \sigma(a)$. Further, recall $A \rtimes G$, and the canonical maps $\iota : A \hookrightarrow A \rtimes G$, $a \mapsto (a, e_G)$, $s : G \to A \rtimes G$, $g \mapsto (e_A, g)$, and $\pi : A \rtimes G \to G$, $(a, g) \mapsto g$.

- Make sure that you know/check (the proofs of the fact) that $A \rtimes G$ is a group, $\iota$ and $s$ are group embeddings, $\pi$ is a group morphism, $s$ is a section of $\pi$, and one has and exact sequence

$$(\mathcal{E}) : 1 \to A \xrightarrow{\iota} A \rtimes G \xrightarrow{\pi} G \to 1.$$ 

- Make sure that you know that if $(\mathcal{E}') : 1 \to A \xrightarrow{\iota'} \Gamma \xrightarrow{\pi'} G \to 1$ is a split exact sequence via $s' : G \to \Gamma$ s.t. $\varphi : G \to \operatorname{Aut}(A)$ is defined via $s'$, then $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic.

- For the semidirect product $\Gamma_0 = A \rtimes G$ defined via the fixed action $\varphi : G \to \operatorname{Aut}(A)$ of $G$ on $A$, and the resulting split extension $\mathcal{E} : 1 \to A \xrightarrow{s} \Gamma_0 \xrightarrow{\pi} G \to 1$, consider/recall:

  - $\mathcal{S}_\mathcal{E} := \{ s : G \to \Gamma_0 | \pi \circ s = \text{id}_G \}$, the space of sections of $\pi : \Gamma_0 \to G$.
  - $Z^1(G, A) := \{ f : G \to A | f(\sigma \tau) = f(\sigma) f(\tau(a)) \}$, set of 1-cocycles of $G$ with values in $A$.
  - $B^1(G, A) := \{ f : G \to A | \exists a \in A \text{ s.t. } f(\sigma) = a^{-1} \sigma(a) \}$, set of 1-coboundaries in $Z^1(G, A)$.
  - The relation $\sim$ on $\mathcal{S}_\mathcal{E}$ defined by $s' \sim s \iff \exists a \in A \text{ s.t. } s'(\sigma) = a^{-1} s(\sigma)a \forall \sigma \in G$.
  - The 1st cohomology set $H^1(G, A) := Z^1(G, A)/\sim$ of $G$ with values in $A$.

- For $A$ abelian, consider all group extensions (E) as above such that the (well defined) action of $G$ on $A$ is given by the fixed $\varphi$. Make sure that you know the (proofs) of the assertions:

  - $B^1(G, A) \subset Z^1(G, A) \subset \text{Maps}(G, A)$, $+$ are subgroups.
  - $H^1(G, A) = Z^1(G, A)/B^1(G, A)$ canonically, hence $H^1(G, A)$ is an abelian group.
  - $(\mathcal{E}) : 0 \to A \xrightarrow{\iota} |A \rtimes G| \xrightarrow{\pi} G \to 1$ is defined by a unique $f_\mathcal{E} \in Z^2(G, A)$ and vice-versa. Moreover: if $f \in Z^2(G, A)$ defines $\mathcal{E}_f$, then $f_{\mathcal{E}_f} = f$ and conversely, $\mathcal{E}_{f_{\mathcal{E}}} = \mathcal{E}$.
  - $B^2(G, A) \subset Z^2(G, A)$, $+$ is a subgroup, and $\mathcal{E}_f \cong \mathcal{E}_{f'}$ iff $f' - f \in B^2(G, A)$.
  - The set of isomorphism of extensions $(\mathcal{E})$ is in bijection with $H^2(G, A) := Z^2(G, A)/B^2(G, A)$.

1) In the above notation, let $s \in \mathcal{S}_\mathcal{E}$ be fixed. Prove the assertions from the class:

   a) $Z^1(G, A) \to \mathcal{S}_\mathcal{E}$, $f \mapsto s_f$ with $s_f(\sigma) = f(\sigma)s(\sigma) \forall \sigma \in G$ is a well-defined bijection.

   b) $\sim$ has as an equivalence relation on $Z^1(G)$.

   c) $s' \sim s$ iff $\exists g \in \Gamma_0$ s.t. $s'(\sigma) = g^{-1}s(\sigma)g$ $\forall \sigma \in G$ iff $\exists f \in B^1(G, A)$ such that $s' = s_f$.

2) Describe the set of splittings $Z^1(G, A)$ and the 1st cohomology set in the following cases:

   a) $1 \to A_n \to S_n \to C_2 \to 1$.

   b) $1 \to C_m \to D_m \to C_2 \to 1$.

   c) $1 \to A \to A \times G \to G \to 1$.

   d) $1 \to A^{[G]} \to A \wr G \to G \to 1$ with $A \wr G$ the wreath product of $A$ and $G$. 

Due: Mo, Nov 20, 2023
3) Let $A$ be abelian, and $G = A^n \rtimes S_n$. Prove/disprove/answer:
   a) $G$ is not perfect, i.e., $[G,G] \neq G$. Further, if $n > 1$, then $[G,G] \neq \{e_G\}$.
   b) $G^{ab}$ is infinite iff $A$ is infinite. Is $G$ solvable for all $n > 0$?

4) Let $G$ be a finite group acting on a finite group $A$ and suppose that the group orders $|G|$ and $|A|$ are relatively prime. Prove/disprove/answer:
   a) If $A$ is solvable, every extension of $G$ by $A$ is split.
   b) Does the same hold if $A$ is not necessarily solvable?

5) Consider the additive groups $\mathbb{Z}$, $\mathbb{Q}$ and $V = (\mathbb{Z}/2, +)^2$. For a finite group $G$, do/answer:
   a) Classify all group extensions $0 \to \mathbb{Q} \to \Gamma_\mathbb{Q} \to G \to 1$, resp. $0 \to \mathbb{Z} \to \Gamma_\mathbb{Z} \to G \to 1$.
   b) Describe all actions of $C_2$ on $V$. Classify all group extensions $0 \to V \to G \to C_2 \to 1$.

6) For $A$ abelian and $f \in Z^2(G,A)$ defining $\mathcal{E}$, i.e., $\mathcal{E} = \mathcal{E}_f$ and $f = f_\mathcal{E}$, prove/disprove:
   \[ \mathcal{E} \] is a split extension if and only if $f \in B^2(G,A)$.
   \textbf{Hint:} Let $\mathcal{E}$ be defined by $f \in B^2(G,A)$, i.e., $f(\rho, \sigma) = f(\sigma) + \rho f(\sigma) - f(\rho \sigma)$ for some $f : G \to A$. Replacing $u_{a\sigma}$ by $u_{a\sigma} = f(\sigma)v_a$, get $u_{a\sigma^2} = f(\rho)u_{a\sigma} \iff f(\rho)v_{a\sigma^2} = f(\rho)v_{a\sigma} 
\text{iff } f(\rho)v_{a\sigma^2} = f(\rho)v_{a\sigma} \iff f(\rho)=(f(\rho) \sigma)v_{a\sigma} \text{iff } v_{a\sigma^2} = v_{a\sigma}$, etc.]

7) In the above notation, prove/answer the assertions from the class:
   a) Every $G$-morphism $f : A \to B$ gives rise to maps $Z^1(f) : Z^1(G,A) \to Z^1(G,B)$ such that $f(B^1(G,A)) \subseteq B^1(G,B)$, hence to a well defined map $H^1(f) : H^1(G,A) \to H^1(G,B)$ \textbf{[HOW]}.
   b) Let $1 \to A \to B \to C \to 1$ be an exact sequence of $G$-groups. Then the \textit{connecting morphism} $\delta^1 : C^G \to H^1(G,A)$ is well defined.
   c) If $B$ above is abelian, the \textit{connecting morphism} $\delta^2 : H^1(G,C) \to H^2(G,A)$ is well defined, and one has a long exact sequence of abelian groups:
   \[ 0 \to A^G \to B^G \to B^G \to H^1(G,A) \to H^1(G,B) \to H^1(G,C) \to H^2(G,A) \to H^2(G,B) \to H^2(G,C). \]

8) Let $G$ and $A$ be finite groups with $n_G = |G|$, $n_A = |A|$ relatively prime. Prove/disprove:
   a) If $A$ is abelian, all extensions of $G$ by $A$ are split.
   b) The same question in the case $A$ is not necessarily abelian.
   \textbf{Hint (a):} Let $\mathcal{E}$ be defined by $f \in Z^2(G,A)$, i.e., $\rho f(\sigma, \tau) + f(\rho, \sigma \tau) = f(\sigma, \tau) + f(\rho \sigma, \tau)$. Setting $a_{\sigma} := \sum f(\sigma, \tau)$, get:
   $\rho(a_{\sigma}) + a_{\rho} = n_G f(\rho, \sigma) + a_{\rho \sigma}$ \textbf{[WHY]}. Hence if $r n_G = 1 \mod n_A$, and $b_{\sigma} = r a_{\sigma}$, get: $b_{\sigma} + \rho(b_{\sigma}) = b_{\rho \sigma} + f(\rho, \sigma)$ \textbf{[WHY]}.
   Proceed as in Problem 6) above by replacing $u_{a\sigma} \text{ by } u_{a\sigma} = b_{\sigma}v_{a\sigma}$, etc. To b): Think harder...