Math 6020 / Problem Set 4 (two pages)

Categories & Functors

Basics

For categories \( C, D, \ldots \), let \( A, A_i, B, B_i, \ldots \) in \( \text{Ob}(C) \) denote objects of \( C, D, \ldots \) and \( f, g, \ldots, \) e.g. \( f \in \text{Hom}_C(A, B) \) denote morphisms. Further, \( I \) is an index set, respectively \( I, \leq \) is a directed (or filtered) ordered set when speaking about projective systems \((A_i, f_{kj})_{i, k \geq j}, \) respectively inductive systems \((B_i, f_{jk})_{i, j \leq k} \) and/or projective (or inverse) limits \( \lim_{\to} A_i \), respectively injective (or direct) limits \( \lim_{\leftarrow} B_i \).

- Be sure that you know the (proofs of the) following:
  - \( \text{id}_A \in \text{Hom}_C(A, A) \) is unique.
  - If \( f : A \to B \) in \( \text{Hom}(A, B) \) is an isomorphism, its inverse \( g \in \text{Hom}(B, A) \) is unique.
  - \( \text{End}_C(A) := \text{Hom}_C(A, A), \circ \) is a monoid w.r.t. composition \( \circ \) of morphisms.
  - \( \text{Aut}_C(A) := \{ f \in \text{End}_C(A) \mid f \text{ is isomorphism} \} \) is a group.

1) Indicate which of the following assertions are true (justify!):
   a) Groups is a full subcategory of Monoids.
   b) Fields is a full subcategory of Rings.
   c) \( F : \text{Rings} \to \text{Ab}, \) by \( R \mapsto R, + \) is a (co/contravariant?) functor.
   d) \( F : \text{Ab} \to \text{Rings}, \) by \( G \mapsto \text{End}(G), +, \circ \) is a (co/contravariant?) functor.

Universal Constructions

2) Complete the proofs of the assertions from the class:
   a) If the product \( \prod_i A_i \) in \( C \) exists, then \( \prod_i A_i \) is unique up to unique isomorphism.
      The same question correspondingly for coproducts \( \coprod_i A_i \).
   b) If \( A \prod B \) exists for all \( A, B \in \text{Ob}(C) \), then finite products \( \prod_{i=1}^n A_i, n > 0 \) exist in \( C \).
      The same question correspondingly for coproducts.

3) Let \( I, \leq \) be a directed ordered set. Complete the proofs of the assertions from the class:
   a) If \( \lim_{\to} A_i \) exists in \( C \), then \( \lim_{\to} A_i \) is unique up to unique isomorphism.
   b) The same correspondingly for \( \lim_{\leftarrow} A_i \).

4) Complete the proofs of the assertions from the class:
   a) If two-fold products and projective limits in \( C \) exist, the \( C \) has arbitrary products.
      Does the converse hold?
   b) Same question about coproducts and colimits.

Basics about categories of b.a.s. (basic algebraic structures)

Let \( C \) be one of the categories of the usual basic algebraic structures we discussed (sets, monoids, (abelian) groups, (commutative) rings, (skew) fields, \( R \)-modules). For an infinite
cardinal $\kappa$, let $C_\kappa$ be the subcategory of $C$ whose objects have cardinality $\leq \kappa$. Complete the proof of the following assertions from the class.

5) Let $X$ be a non-empty set with $|X| \leq \kappa$. Prove/disprove the following:
   a) The collection $A$ of all the monoid/group-ring/(skew) field structures on $X$ is a set.
   b) The collection $M$ of all the $R$-module structures on $X$ with $R \in A$ is a set.
   c) Give an estimate in terms of $\kappa$ for the cardinalities $|A|$ and $|M|$.

6) (Using Problem 5 above) Prove/disprove/answer the following:
   a) $C_\kappa$ is a full subcategory of $C$.
   b) $C_\kappa$ is equivalent to a small subcategory $C^0_\kappa$ of $C_\kappa$.
   c) Recalling which products exist in $C$, which products exist in $C_\kappa$?

7) Let $(A_i, f_{kj})_{i, k \geq j}$ and a projective system and $(B_i, g_{jk})_{i, j \leq k}$ and an inductive system in $C$. Complete the proofs of the assertions from the class:
   a) Viewing $C$ as a subcategory of $\text{Sets}$, prove that in $\text{Sets}$ one has:
      - The projective limit of $(A_i, f_{kj})_{i, k \geq j}$ is $\varprojlim A_i := \{(x_i)_i \in \prod_i A_i \mid f_{kj}(x_k) = x_j \forall k \geq j\}$.
      - The inductive limit of $(B_i, g_{jk})_{i, j \leq k}$ is $\varinjlim B_i := \prod_i B_i/\sim$, where $\sim$ the equivalence relation on $\prod_i B_i$ generated by $(j, x_j) \sim (k, x_k) \iff f_{jk}(x_j) = x_k \forall j \leq k$.
   b) $\varprojlim A_i$ and $\varinjlim B_i$ defined above carry canonically composition laws making them objects of the category $C$, and $f_i : \varprojlim A_i \to A_i$ and $g_i : B_i \to \varinjlim B_i$ from $\text{Sets}$ are in $\text{Mor}(C)$.
   c) Finally, viewing $\varprojlim A_i$ and $\varinjlim B_i$ as objects in $C$, one has:
      - $\varprojlim A_i$ is the projective limit of $(A_i, f_{kj})_{i, k \geq j}$ in $C$.
      - $\varinjlim B_i$ is the injective limit of $(B_i, f_{jk})_{i, j \geq k}$ in $C$. 

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