Math 371 / Problem Set 7 (one page)

More about of Rings/Modules

1) For a square free \( d \in \mathbb{Z} \), consider \( R = \mathbb{Z}[\sqrt{d}] = \{ a + b\sqrt{d} | a, b \in \mathbb{Z} \} \). Prove the following:
   a) \( R \) is a subring of \( \mathbb{R} \) and \( f_R : R \to R, \alpha = a + b\sqrt{d} \mapsto a - b\sqrt{d} = \overline{\alpha} \) is a ring isomorphism.
   b) The fraction field of \( R \) is \( F = \{ x + y\sqrt{d} | x, y \in \mathbb{Q} \} \) and \( f_F : F \to F, \alpha = x + y\sqrt{d} \mapsto x - y\sqrt{d} = \overline{\alpha} \) is a field isomorphism.
   c) \( \alpha \in R \) is a unit iff \( \alpha \overline{\alpha} = \pm 1 \).

2) Let \( R \) be a principal ideal domain, and \( \Sigma \subset R \) be a multiplicative system. Prove/disprove:
   a) The ring of fractions \( R_\Sigma = \Sigma^{-1} R \) is a principal ideal domain.
   b) \( R_\Sigma \) is a local ring, i.e., it has a unique maximal proper ideal iff there is a \( \pi \in R \) prime element such that \( \Sigma = \{ r \in R | \pi \nmid r \} \).

4) Prove/disprove/answer the following:
   a) The polynomial ring \( R[t] \) over a commutative ring \( R \) is PID iff \( R \) is a field.
   b) The ring of continuous functions \( C(I, \mathbb{R}) \) with \( I = [a, b] \subset \mathbb{R} \) is not PID.

5) Prove that \( \sim_\Sigma \) is an equivalence relation on \( \mathcal{M} \). Let \( \frac{x}{r} := (x, r) \sim \) denote the equivalence classes and set \( M_\Sigma := \Sigma^{-1} = \{ \frac{x}{r} | x \in M, r \in \Sigma \} \).

   Define on \( M_\Sigma \) an addition and an action (or outer multiplication) of \( R_\Sigma \) by
   \[
   \frac{x}{r} + \frac{y}{s} = \frac{rx + sy}{rs}, \quad a \cdot \frac{x}{r} = \frac{ax}{rs}.
   \]

   Note that in particular, \( R \) also acts on \( M_\Sigma \) through the canonical map \( \iota : R \to R_\Sigma, a \mapsto \frac{a}{1} \), that is, via \( a \cdot \frac{x}{s} = \frac{ax}{s} \).

6) Prove the following:
   a) The \( + \) in \( M_\Sigma \) and action of \( R, R_\Sigma \) are well defined and make \( M_\Sigma \) into \( R, R_\Sigma \) module.
   b) \( \iota_\Sigma : M \to M_\Sigma, x \mapsto \frac{x}{1} \) is \( R \)-morphism. \( \text{Ker}(\iota_\Sigma) = \{ 0_M \} \) if \( r \cdot x = 0_M \Rightarrow x = 0_M \forall r \in \Sigma \).
   c) If \( f : N \to M \) is a morphism of \( R \)-modules, then \( f_\Sigma : N_\Sigma \to M_\Sigma, \frac{x}{r} \mapsto \frac{f(x)}{r} \) is a morphism of \( R_\Sigma \) and \( R \) modules.

   Recall that for every prime ideal \( \mathfrak{P} \in \text{Spec}(R), \Sigma_\mathfrak{P} := R \setminus \mathfrak{P} \) is a multiplicative system, see HW 6, Problem 1. Let \( R_\mathfrak{P} \) and \( M_\mathfrak{P} \) be the corresponding ring/module of fractions.

7) In the above notation prove/disprove/answer:
   a) \( M = \{ 0_M \} \) is the trivial \( R \)-module iff \( M_\mathfrak{P} = \{ 0_M \} \) is the trivial \( R_\mathfrak{P} \)-module for all \( \mathfrak{P} \).
   b) \( f : N \to M \) is injective/surjective/isomorphism iff \( f_\mathfrak{P} : N_\mathfrak{P} \to M_\mathfrak{P} \) is so for all \( \mathfrak{P} \).
   (●) Does the same hold if one replaces prime ideals by maximal ideals?