Math 371 / Problem Set 4 (one page)

Miscellaneous (Groups & Rings)

1) Let \( \alpha := \sqrt{2} \in \mathbb{R} \) and \( R[\alpha] := \{ a + b\alpha + c\alpha^2 \in \mathbb{R} \mid a, b, c \in R \} \). Prove/disprove:
   a) If \( R = \mathbb{Z} \), then \( \mathbb{Z}[\alpha] \) is a subring of the field of real number \( \mathbb{R} \).
   b) If \( R = \mathbb{Q} \), then \( \mathbb{Q}[\alpha] \) is a subfield of the field of real numbers \( \mathbb{R} \).
   c) One has that \( \mathbb{Q}[\alpha] = \text{Quot}(\mathbb{Z}[\alpha]) \) is the field of fractions of \( \mathbb{Z}[\alpha] \).

2) Let \( G \) be a group, and \( \Delta_1, \Delta_2 \triangleleft G \) be normal subgroups, and considering the factor groups \( \overline{G_1} = G/\Delta_1, \overline{G_2} = G/\Delta_2 \), define \( \varphi : G \to \overline{G_1} \times \overline{G_2} \) by \( g \mapsto (g\Delta_1, g\Delta_2) \). Prove/disprove:
   a) \( \varphi \) is a group homomorphism with \( \text{Ker}(\varphi) = \Delta_0 := \Delta_1 \cap \Delta_2 \).
   b) \( G' := \Delta_1\Delta_2 := \{ g_1g_2 \mid g_1 \in \Delta_1, g_2 \in \Delta_2 \} \triangleleft G \) is a normal subgroup of \( G \).
   c) \( \varphi \) is surjective iff \( G' = G \).

For Problem 3 below, recall/compare with HW # 3, Problems 6, 7).

3) Let \( R \) be a ring, \( I_1, I_2 \subset R \) be ideals, and for the resulting factor rings \( \overline{R_1} := R/I_1, \overline{R_2} := R/I_2 \), consider \( \varphi : R \to \overline{R_1} \times \overline{R_2}, r \mapsto r + I_1, r + I_2 \). Prove/disprove:
   a) \( \varphi \) is a ring morphism with \( \text{Ker}(\varphi) = I_0 := I_1 \cap I_2 \).
   b) \( I' := I_1 + I_2 \subset R \) is an ideal of \( R \).
   c) \( \varphi \) is surjective iff \( I' = R \) iff \( I_1, I_2 \) are comaximal, i.e., \( 1_R \in I_1 + I_2 \).

Terminology. Assertion c) above is the generalized Chinese Remainder Thm (for \( R \)).

To think about: What would/should be the generalized Chinese Remainder Thm for:
   (i) Three numbers ideals \( I_1, I_2, I_3 \subset R \); (ii) Finitely many ideals \( I_1, \ldots, I_n \subset R \).

4) Let \( R = R_1 \times R_2 \) be the product of the rings \( R_1, R_2 \). Prove/disprove:
   a) All the ideals of \( I \) are of the form \( I = I_1 \times I_2 \) with \( I_1 \subset R_1, I_2 \subset R_2 \) ideals.
      What is the corresponding assertion for left/right ideals?
   b) If \( I = I_1 \times I_2 \) then \( R/I \) is (canonically) isomorphic to \( R_1/I_1 \times R_2/I_2 \).

5) In the context form Problem 4) above, prove/disprove:
   a) \( I \) is principal (i.e., of the form \( I = Rr \) with \( r \in R \)) iff \( I_1 \subset R_1, I_2 \subset R_2 \) are principal.
   b) \( I \) is prime ideal iff either \( I = R_1 \times I_2 \) or \( I = I_1 \times R_2 \) with \( I_1, I_2 \) prime ideals.
   c) \( I \) is maximal ideal iff either \( I = R_1 \times I_2 \) or \( I = I_1 \times R_2 \) with \( I_1, I_2 \) maximal ideals.

6) Let \( \mathcal{P}(X), \Delta, \cap \) be the ring defined in HW #2, Problem 1), and for an arbitrary ring \( R \), let \( \text{Maps}_0(X, R) := \{ f \in \text{Maps}(X, R) \mid f(x) = 0_R \text{ f.a.a. } x \in X \} \). Prove/disprove/answer:
   a) Show that \( \varphi : \mathcal{P}(X) \to \text{Maps}(X, \mathbb{F}_2), A \mapsto \chi_A \) is a ring isom, where \( \chi_A : X \to \mathbb{F}_2 \) is the characteristic function of \( A \subset X \), i.e., \( \chi_A \) is defined by \( \chi_A(x) = 1_{\mathbb{F}_2} \text{ iff } x \in A \).
   b) \( \text{Maps}_0(X, R) \) is a free \( R \)-submodule of the \( R \)-module \( \text{Maps}(X, R) \).
   c) Show that \( \text{Maps}_0(\mathbb{N}, R) \) and \( R[t], + \) are isomorphic \( R \)-modules.

• Note: It is an open problem whether \( \text{Maps}(X, \mathbb{Z}) \) is always a free \( \mathbb{Z} \)-module(!)