# **Math 702** (Spring 2022)

## • About this course

This course is an introduction to methods, tools, and open problems in Valuation Theory and its applications algebraic/arithmetic geometry, e.g. local-global principles for the Brauer group and higher (Galois) cohomology groups, effectiveness in arithmetic geometry, anabelian geometry (e.g. the section conjecture, anabelian reconstruction). The prerequisites are a solid knowledge of algebra, especially commutative algebra, and basics of topology, scheme theory (algebraic geometry). [OTOH, working a little bit harder, every grad student can handle the prerequisites as we go.] The course will be partly expository, and partly proofs will be given (either detailed, or just sketchy). Most topics are suitable to be presented by the participants (it is highly recommended that you consider that; those interested should let me know what topics are they most interested in / would like to learn more about).

Below is a (not comprehensive) **list of topics** which could be discussed/touched upon:

**NOTE**: The main sources are in **boldface**, articles are numbered.

## • General Valuation Theory (quick review)

([BOU], Ch. 6, [Z-S], Ch. 6, [Kh], [En], [E-P], [Ri], Google it!)

1) Basics of Valuation rings, Places, Valuations (Google it!)
    - Valuation rings, Places, Valuations; Rank
    - Examples, Local Desingularization Problem
    - Refinements/Coarsenings, Discrete
    - Riemann-Zariski space $\mathbf{Val}_\Sigma(K)$
    - Approximation of valuation for $K$ "arithmetical"
2) Prolongation of Valuations ([BOU], Ch. 6, [Kh], Ch. 6, 11, Google it!):
    - Existence of Valuations (Chevalley's Thm)
    - The canonical projection $\mathbf{Val}_\Sigma(L) \twoheadrightarrow \mathbf{Val}_\Sigma(K)$ for $L|K$
    - The fundamental (in)equality, Defect.
    - Scheme theoretical description of $\mathbf{Val}_\Sigma(K)$
3) Hilbert Decomposition Theory ([BOU], Ch. 5, [Se2], [Kh], Ch. 6-9, Google it!)
    - Hilbert Decomposition/Ramification Theory
    - Henselization
    - Approximation of Valuations revisited: The maps $\mathbf{Val}(\overline{K}) \to \mathbf{Sg}(G_K), \overline{v} \mapsto D_{\overline{v}}, I_{\overline{v}}, V_{\overline{v}}$
        - I) Chebotarev Density Thm (CDT) and $\mathbf{Val}(K)$
        - II) Higher dim variants of CDT

*Possible Supplements* (very) short surveys on:

- (Local) Desingularization / Alterations ([K-K, Tk1, Tk2], [dJ1, dJ2, ILO], Exp. X, [C-P], . . . )
- Higher dim CDT and Holschbach's Thm [Se3, Ho]

## • AKE Principles

([B-S, Ax, A-K1, A-K2, K-P], ...)

1) Basics about $C_i$ fields (e.g. [Pf1], Google it!)
   - Definitions, Examples, Properties
   - Artin's Conjecture / Terjanian's Counterexample
   - Ax–Kochen point of view: What about the family $(\mathbb{Q}_p)_p$, is it $C_2$?
2) Basics about ultraproducts/ultrapowers ([B-S, Ch], Google it!)
   - Definitions, Properties, Examples
   - The Ax-Kochen-Ershov (AKE) Principle.
   - Real/$p$-adically closed fields
   - **Fact.** $\prod_p \mathbb{F}_p((t))/\mathcal{U} \cong \prod_p \mathbb{Q}_p$, Consequences for $(\mathbb{Q}_p)_p$
   - Further Applications

(∗) **Big$^3$ Open Problem:** What about char $> 0$, e.g. is there an AKE for $\mathbb{F}_p((t))$?

*Possible Supplement*: (very) short surveys on:
- The case of tame fields ([K-P], ...)
- Applications to the Collio-Thélène Conjecture
- The class of maximal valued fields ([AKP, A-K], ...)

## • LGP (I) & (Birational) $p$-adic Grothendieck SC

([Ta, Ro, Li, P1], [Ko1, St2], [P2, P3], [P-S], ...)

1) The Brauer group of a smooth variety
   - Generalities Bruer groups, the unramified part of the Brauer group Examples, $\mathrm{Br}\big(k(t)\big)$ vs $\mathrm{Br}(\mathbb{P}^1_k)$, etc.
   - Tate's Duality Thm for curves over $p$-dic fields, Roquette, Lichtenbaum
   - LGP for $\mathrm{Br}(K)$, $\mathrm{td}(K|k) = 1$, $k|\mathbb{Q}_p$ algebraic
2) Grothendieck SC/Variants ($p$-adic, birational, etc.)
   - The generic case [Ha, Wa]/the "no points" cases [H-S, St1], ...
   - The (minimalistic) $p$-adic birational SC [Ko1, St2, P2, P2], ...
   - Partial results for the $p$-adic SC [Mz1, Mz2, P-S], ...

*Possible supplements* (very) short survey on::
- Conditional partial results in the global case [**?**, **?**], ...

(∗) **Big$^3$ Open Problem:** Show that $G_K \cong G_K$ arises via Galois theory ...

## • LGP (II) & f.o. effectiveness in arithmetic geometry

1) **Detours**:
   - Quadratic forms, Witt ring of anisotropic quadratic forms, Pfister formas
   - Minor K-Theory
   - Milnor/Bloch-Kato Conjectures

- Cohomological dim of fields, e.g. finitely generated field
- $\mathrm{cd}_2(F)$ is f.o., and so is $\mathrm{Kr.dim}(K)$ for finitely generated fields

2) Kato's (conjectural) higher LGPs of Hasse type
   - Survey of known facts: **Kato, Jannsen, Kerz–Saito, . . .** [Ka, K-S, Jn]

3) **First order** facts about finitely generated fields $K$
   - $\mathrm{char}(K) = 0$ is f.o.
   - $K^{\mathrm{abs}} \subset K$ is f.o.
   - geometric prime divisors $\mathcal{D}^1_K$ are f.o.
   - Isomorphism type of $K$ is f.o.

(∗) **Open Problem:** What about other "arithmetically interesting" fields?

## • Valuations via Galois theory & Anabelian geometry

([Ne, P1, Ko1, To1], . . . )

1) Recovering valuations from Galois theory
   - The Neukirch–Uchida Thm, e.g. [NSW], **XII**
   - The higher dimensional case
2) Grothendieck's Anabelian Conjectures for Curves (survey)
   - Recovering points of curves over finite fields
   - The case of affine hyperbolic curves over finite fields
   - The case of affine hyperbolic curves (Specialization techniques)
   - Mochizuki's Thm over sup-$p$-adic field.
3) Anabelian Phenomena over algebraically closed fields (**survey**)
   - Bogomolov's Program (BP)
   - Refinements of (BP)

(∗) **Open Problem:** Distinguish the prime divisors among the quasi-prime divisors.

## References


[A-K] Anscombe, S. and Kuhlmann, F.-V.. *Notes on extremal and tame valued fields,* J. Symb. Logic **81** (2016), 400–416. (document)

[AEJ] Arason, J., Elman, R. and Jacob, B., *Rigid elements, valuations, and realization of Witt rings,* J. Algebra **110** (1987), 449–467.

[AKNS] Aschenbrenner, M., Khélif, A., Naziazeno, E. and Scanlon, Th., *The logical complexity of finitely generated commutative rings,* Int. Math. Res. Notices (to appear).

[Ax] Ax, J., *The elementary theory of finite fields,* Annals of Math. **88** (1968), 239–271. (document)

[A-K1] Ax, J. and Kochen, S., *Diophantine problems over local fields* I, Amer. J. Math. **87**, (1965), 605–630 (document)

[A-K2] Ax, J. and Kochen, S., *Diophantine problems over local fields* III. *Decidable fields,* Annals of Math. **83** (1966), 437–456. (document)

[AKP] Salih Azgin, Franz-Viktor Kuhlmann and Florian Pop, *Characterization of Extremal Valued Fields,* Proc. AMS, Vol. **140** (2012), 1535–1547. (document)

[B-S] Bell, J. Land Slomson, A. B., Models and Ultraproducts: An Introduction; Dover Books on Math. (document)

[Bo] [Bo] Bogomolov, F. A., *On two conjectures in birational algebraic geometry,* in: Algebraic Geometry and Analytic Geometry, ICM-90 Satellite Conference Proceedings, eds A. Fujiki et all, Springer Verlag Tokyo 1991.

[B-T1] Bogomolov, F. A. and Tschinkel, Y., *Commuting elements in Galois groups of function fields,* pp. 75-120; in: Motives, Polylogarithms and Hodge theory", eds F.A. Bogomolov, L. Katzarkov, International Press, 2002.

[BOU] Bourbaki, Algèbre commutative, Hermann Paris 1964. (document)

[C-F] Cassels & Fröhlich, *Algebraic Number Theory.*

[Ch] Chatzidakis, Z., *Notes on the model theory of finite and pseudo-finite fields,* **Notes** #16. See `https://www.math.ens.fr/~zchatzid/` (document)

[C-P] Cossart, V. and Piltant, O., *Resolution of singularities of threefolds in positive characteristic II,* J. Algebra **321** (2009), 1836–1976. (document)

[Ef] Ido Efrat, Valuations, Orderings and Milnor $K$-Theory, AMS Mathematical Surveys and Monographs, Vol. **124**, 2006.

[EKM] Elamn, R., Karpenko, N., and Merkurjev, A., The Algwbraic and geometric Theory of Quadratic Forms, AMS Colloquium Publ, Vol. **56**, 2008.

[En] Endler, O., *Valuation Theory,* Springer Verlag. (document)

[E-P] Engler, A. J. and Prestel, A., Valued Fields, Springer Monographs in Mathematics Series, Springer-Verlag Berlin and Heidelberg GmbH & Co. KG, 2005. (document)

[F-J] Fried, M. and Jarden, M., Field Arithmetic (third revised edition). Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge; Springer Verlag, ISSN: 0071-1136.

[GPR] Green, B.-W., Pop F. and Roquette, P. *On Rumely's LGP,* Jahresbericht DMV, Band **97** (1995), 43–74.

[Ha] Hain, R., *Rational points of universal curves,* See arXiv:mathNT/1001.5008v1. (document)

[H-H] David Harbater and Julia Hartmann, *Patching over fields,* Israel J. Math. **176** (2010), 61–107. See `arXiv:0710.1392`.

[Ho] Hoshi, Y., *Existence of nongeometric pro-p Galois sections of hyperbolic curves,* preprint, January 2010, RIMS preprint1689. (document)

[H-S] Harari, D., Szamuely, T., *Galois sections for abelianized fundamental groups,* Appendix by E. V. Flynn, Math. Ann. **344** (2009), 779–800. (document)

[Ho] Holschbach, A., *A Chebotarev-type density theorem for divisors on algebraic varieties,* arXiv:1006.2340v2 [math.AG] Wed, 19 Sep 2012. (document)

[ILO] Illusie, L. and Laszlo, Y. and Orgogozo, F., *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents,* Séminaire à l'École Polytechnique 2006–08, See `arXiv:1207.3648v1 [math.AG], July 2012` (document)

[Jn] Jannsen, U., *Hasse principles for higher-dimensional fields,* Annals of Math. **183** (2016), 1–71. (document)

[dJ1] de Jong, J., *Smoothness, semistability and alterations,* IHES Publ. Math. **83** (1996), 51-93. (document)

[dJ2] de Jong, J., *Families of curves and alterations,* Ann. Inst. Fourier (Grenoble) **47** (1997), no. 2, 599Ð621. (document)

[Ka] Kato, K., *A Hasse principle for two dim global fields*, J. reine angew. Math. **366** (1986), 142–180. (document)

[K-S] Kerz. M. and Saito, Sh., *Cohomological Hasse principle and motivic cohomology for arithmetic schemes,* Publ. Math. IHES **115** (2012), 123–183. (document)

[K-K] Knaf, H. and Kuhlmann, F.-V., *Abhyankar places admit local uniformization in any characteristic,* Ann. Scient. ENS **38** (2005), 833–846. (document)

[Ko1] Koenigsmann, J., *On the 'section conjecture' in anabelian geometry,* J. reine angew. Math. **588** (2005), 221–235. (document)

[Ko2] Koenigsmann, J., *Defining transcendentals in function fields,* J. Symbolic Logic **67** (2002), 947–956.

[Kh] Kuhlmann, F.-V., *Book on Valuation Theory,* see https://math.usask.ca/ fvk/Fvkbook.htm (document)

[K-P] Kuhlmann, F.-V. - Pal, K., *The model theory of separably tame fields,* J. Alg. **447** (2016), 74–108. (document)

[Kl] Kollar, J. *Rationally connected varieties over local fields,* Annals of Math. **150** (1999), 357–367.

[Li] Lichtenbaum, S., *Duality theorems for curves over p-adic fields,* Inventiones Math. **7** (1969), 120–136. (document)

[M-S] Merkurjev, A. S. and Suslin, A. A., *K-cohomology of Severi–Brauer variety and norm residue homomorphism,* Math. USSR Izvestia **21** (1983), 307–340.

[Mi] Milne, J. *Étale Chomology,* See `http://www.jmilne.org/math/Books/ECpup1.pdf`

[Mz1] Mochizuki, Sh., *The local pro-p anabelian geometry of curves,* Invent. Math. **138** (1999), 319–423. (document)

[Mz2] Mochizuki, Sh., *Topics surrounding the anabelian geometry of hyperbolic curves,* in: Galois groups and fundamental groups, MSR Publ. **41**, Cambridge Univ. Press (2003), 119–165. (document)

[MB] Moret-Bailly, L. *Groupes de Picard et problèmes de Skolem I & II,* Ann. Scient. ÉNS (4 ser.) **22** (1989), 161–179 & 181–194.

[Ne] Neukirch, J., *Kennzeichnung der p-adischen und endlichen algebraischen Zahlkörper,* Inventiones Math. **6** (1969), 269–314. (document)

[NSW] Neukirch, J, Schmidt, A. and Winberg, K., *Cohomology of Number Fields,* 2nd Edition, 2015. (document)

[Pf1] Pfister, A., Quadratic Forms with Applications to Algebraic Geometry and Topology, LMS LNM **217**, Cambridge University Press 1995; ISBN 0-521-46755-1. (document)

[Pf2] Pfister, A., *On the Milnor conjectures: history, influence, applications,* Jahresber. DMV **102** (2000), 15–41.

[Po1] Poonen, B., *Uniform first-order definitions in finitely gen. fields,* Duke Math. J. **138** (2007), 1–21.

[P-P] Bjorn Poonen and Florian Pop, *First-order definitions in function fields over large fields,* in: *Model Theory with applications to algebra and analysis,* LMS LNS Series, Vol. **350**, eds Chatzidakis, Macpherson, Pillay, Wilkie. Cambridge University Press, Cambridge 2008; pp. 255–272.

[P1] Pop, F. *Galoissche Kennzeichnung p-adisch abgeschlossener Körper,* J. reine angew. Math. 392 (1988), 145–175. (document)

[P2] Pop, F. *On the birational p-adic section Conjecture,* Compositio Math. **146** (2010), 621–637. (document)

[P3] Pop, F. *$\mathbb{Z}/p$ metabelian birational p-adic section conjecture for varieties,* Compositio Math. **153** (2017), 1433–1445. (document)

[P2] Pop, F. *Anabelian Phenomena,* in: *Non-abelian Fundamental Groups and Iwasawa Theory,* eds: Coates, Kim, Pop, Saidi, Schneider; LMS LNS **393**, Cambridge Univ. Press 2011; pp. 1–55. (document)

[P-S] Pop, F. and Stix, J., *Arithmetic in the fundamental group of a p-adic curve –On the p-adic section conjecture,* J. reine angew. Math. **725** (2017), 1–40. (document)

[Ri] Ribenboim, P., *The Theory of Classical Valuation,* Springer Verlag... (document)

[P-R] Prestel, A. and Roquette, P., *Formally p-adic fields,* LNM 1050, Springer Verlag 1985...

[Ro1] Robinson, Julia, *Definability and decision problems in arithmetic,* J. Symb. Logic **14** (1949), 98–114.

[Ro2] Robinson, Julia, *The undecidability of algebraic fields and rings,* Proc. AMS **10** (1959), 950–957.

[Ro] Roquette, P., *Splitting of algebras by function fields of one variable,* Nagoya Math. J. **27** (1966), 625–642. (document)

[Ru] Rumely, R., *Undecidability and Definability for the Theory of Global Fields*, Transactions AMS **262** No. 1, (1980), 195–217.

[Sc] Scanlon, Th., *Infinite finitely generated fields are biinterpretable with* **N**, JAMS **21** (2008), 893–908. *Erratum,* J. Amer. Math. Soc. **24** (2011), p. 917.

[Se1] Serre, J.-P. *Cohomologie Galoisienne,* LNM 5, Springer 1965.

[Se2] Serre, J.-P. *Local fields,* Springer 1965. (document)

[Se3] Serre, J.-P. *Zeta and L-functions,* in: Arithmetical Algebraic Geometry, Proc. Conf. Purdue 1963 (1965), 82-92. (document)

[St1] Stix, J., *On the period-index problem in light of the section conjecture,* Am. J. Math. **132** (2010), 157–180. (document)

[St2] Stix, J., *Birational p-adic Galois sections in higher dimensions,* Israel J. Math. **198** (2013), 49–61. (document)

[Ta] Tate, J. *Cohomology of abelian varieties over p-adic fields,* Notes by Serge Lang, Princeton University, May 1959. (document)

[Tk1] Temkin, *Inseparable local uniformization,* J. Algebra **373** (2013), 65–119. (document)

[Tk2] Temkin, M., *Tame distillation and desingularization by p-alterations,* Ann. of Math. **186** (2017), 97–126. (document)

[To1] Topaz, A., *Commuting-Liftable Subgroups of Galois Groups II,* J. reine angew. Math. (in print). See `http://arxiv.org/abs/1208.0583` (document)

[To2] Topaz, A. *A survey of almost abelian anabelian geometry,* BIRS Workshop, June 2017 See `https://www.birs.ca/events/2017/5-day-workshops/17w5112/videos`

[Wa] Watanabe, T. *On the sections of universal hyperelliptic curves,* See arXiv:1609.04124v1 [math.AG] (document)

[Z-S] Zariki. P and Samuel, P., *Commutative Algebra,* Vol. II, Springer-Verlag, New York, 1975. (document)