Due: Fr, Apr 8, 2022

Math 314 / Problem Set 9 (two pages)

• **Study/read:** *Spaces with inner product*

Ch. 5-6 *LADW* by Treil; Ch. 8 *Linear Algebra* by Hoffman & Kunze.

(!) Make sure that you understand perfectly the definitions, examples, theorems, etc.

• **Basics about** $\mathbb{C} = \mathbb{R}+i\mathbb{R}$. For $z = a+bi \in \mathbb{C}$, define: real/imaginary parts $\mathfrak{Re}(z) = a$, $\mathfrak{Im}(z) = b$, complex conjugate $\overline{z} := a - bi$, abs. value $|z| \overset{\text{Def}}{=} \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$. Answer/prove:

a) Compute $(3 + 2i)(5 - 3i)$, $(2 - 3i)/(1 - 2i)$, $\mathfrak{Re}\big((2 - 3i)/(1 - 2i)\big)$, $(1 + i)^k$ in $\mathbb{C}$.

b) $z \mapsto \overline{z}$ satisfies: (i) $\overline{(z' + z)} = \overline{z}' + \overline{z}$, $\overline{z'z} = \overline{z}'\,\overline{z}$; (ii) $\overline{az} = a\overline{z}\ \forall\, a \in \mathbb{R}$, $z, z' \in \mathbb{C}$.

c) $|z + z'| \le |z| + |z'|$, $|z'z| = |z'|\,|z|\ \forall\, z', z \in \mathbb{C}$.
$|z' + z| = |z'| + |z|$ iff there exists $a \in \mathbb{R}$, $a > 0$ such that $z' = az$ or $z = az'$.

1) For $F = \mathbb{R}$ or $F = \mathbb{C}$, $v \in V := F^m$, recall $\|v\|_\infty$, $\|v\|_p$ for $1 \leqslant p < \infty$ as defined in class.

a) Draw the unit balls $B(0, 1) := \{v \in \mathbb{R}^2 \,\big|\, \|v\|_p \le 1\}$ for $p = 1$, $p = 2$, $p = 4$.

b) Prove/disprove: The balls $B_r(v) := \{v' \in V \,\big|\, \|v' - v\|_p \leqslant r\}$ are convex.

c) Prove/disprove: $\|v\|_q \leqslant \|v\|_p$ for $p \leqslant q$, and $\|v\|_\infty := \lim\limits_{p\to\infty} \|v\|_p$?

2) Which of the following define an inner product on $\mathbb{R}^2$, resp. $\mathbb{C}^2$ (justify if not obvious):

(i) $\big((a_1, a_2)\,|\,(b_1, b_2)\big) := a_1b_1 - a_2b_2$; (ii) $\big((a_1, a_2)\,|\,(b_1, b_2)\big) := a_1b_1 + a_1b_2 + a_2b_1 + 3a_2b_2$

(i) $\big((z_1, z_2)\,|\,(z_1', z_2')\big) := z_1z_1' + z_2z_2'$; (ii) $\big((z_1, z_2)\,|\,(z_1', z_2')\big) := |z_1z_1'| + |z_2z_2'|$

3) Let V be a $\mathbb{C}$-v.s. with inner product $\big(\,|\,\big)$ and $a, b, c, d \in \mathbb{C}$, $u, v \in V$. Given that $\|u\| = 2$, $\|v\| = 3$, $\big(u\,|\,v\big) = 2 + i$, compute: $\|u + 2v\|$, $\|u - v\|$, $\big(u + v\,|\,u - iv\big)$, $\big(au + bv\,|\,cu + dv\big)$.

• **From now on** let $F = \mathbb{R}$ or $F = \mathbb{C}$, and V be an F-vector space.

4) Let $\|\ \|$ be defined by $\big(\,|\,\big)$ on V, and $u, v \in V$, $v \neq 0_V$. Prove/disprove/answer:
(i) $\|u \pm v\|^2 = \|u\|^2 + \|v\|^2 \pm 2\,\mathfrak{Re}\big(u\,|\,v\big)$. (ii) $\|v + u\| = \|v\| + \|u\|$ iff $\exists\, a \in \mathbb{R}_{\geqslant 0}$ s.t. $u = av$.

5) Prove in all detail the following assertions from the class:

a) First, *Polarization Identity* (PoI) and the *Parallelogram Identity* (PaI) mentioned in class. Second, a norm $\|\ \|$ satisfies (PaI) if $\|\ \|$ is defined by some $\big(\,|\,\big)$.

b) If $(v_i)_i$ generate V and $u, u' \in V$, then $u = 0_V$ iff $\big(u\,|\,v\big) = 0\ \forall\, v \in V$ iff $\big(u\,|\,v_i\big) = 0\ \forall\, v_i$. Second, $u = u'$ iff $\big(v\,|\,u\big) = \big(v\,|\,u'\big)\ \forall\, v \in V$ iff $\big(v_i\,|\,u\big) = \big(v_i\,|\,u'\big)\ \forall\, v_i$.

c) Let $\mathcal{A} = (v_1, \dots, v_n)$ of V be a basis of V, e.g. $\mathcal{E}$ of F^m or $\mathcal{E}_{m\times n}$ of $F^{m\times n}$. Then $\big(\,|\,\big)_{\mathcal{A}}$ as defined in class is an inner product on V. And if $\mathcal{A} = \mathcal{E}_{m\times n}$, then $\big(A\,|\,B\big)_{\mathcal{A}} = \mathrm{Tr}(B^*A)$.

d) For $F = \mathbb{R}$ and $\mathcal{A} = \big((1, 1), (\text{-}2, \text{-}1)\big)$, draw the unit ball in $V = \mathbb{R}^2$ w.r.t. $\big(\,|\,\big)_{\mathcal{A}}$.

6) Let V be an F-v.s. endowed with $(\,|\,)$. True of false (justify if not obvious):

a) An orthogonal/orthonormal system $\mathcal{A} = (v_i)_{i \in I}$ with $v_i \in V$ is linearly independent.

b) If $\mathcal{B} = (w_n)_{n\in\mathbb{N}}$ is a basis of V, there exists an orthonormal basis $\mathcal{A} = (v_n)_{n\in\mathbb{N}}$ of V such that $\langle w_1, \dots, w_n\rangle_F = \langle v_1, \dots, v_n\rangle_F$ for each $n \in \mathbb{N}$.

c) For a system of vectors $\mathcal{A} = (v_1, \dots, v_n)$ with $v_i \in V$ the following are equivalent:

  i) $\mathcal{A}$ is an orthonormal basis of V.

  ii) For every $v \in V$ one has: $v = (v|v_1)v_1 + \cdots + (v|v_n)v_n$.

  iii) For every $v \in V$ one has $\|v\|^2 = |(v|v_1)|^2 + \cdots + |(v|v_n)|^2$.

7) Let V be an F-v.s. endowed with $(\,|\,)$, and $\mathcal{A} = (v_1, \dots, v_n)$ generate V. Prove/disprove:

a) $\mathcal{A}$ is orthonormal iff for all $v = \sum_i a_i v_i, w = \sum_i b_i v_i \in V$ one has $(v\,|\,w) = \sum_i a_i \bar{b}_i$.

b) If $\mathcal{A}$ is orthonormal, the Parseval's identity holds:

$$(v\,|\,w) = (v\,|\,v_1)\overline{(w\,|\,v_1)} + \cdots + (v\,|\,v_n)\overline{(w\,|\,v_n)}.$$

8) For $F = \mathbb{R}$, let $V := \langle\, (1,1,1,1), (1,2,3,4), (\text{-}1,0,1,2)\rangle \subset \mathbb{R}^4$.

a) Find an orthogonal basis of $V \subset \mathbb{R}^4$ using the standard inner product $(\,|\,)$ on $\mathbb{R}^4$.

b) Find an orthogonal basis of $V^\perp \subset \mathbb{R}^4$ using the standard inner product $(\,|\,)$ on $\mathbb{R}^4$.

9) Lat $F = \mathbb{R}$. Answer the following concerning $V = \mathbb{R}^4$:

a) Given $V = \langle(1,3,1,1), (2,\text{-}1,1,0)\rangle$, find the orthogonal projection of $(1,1,1,1)$ onto V.

b) Given $W = \langle(1,\text{-}1,1,0), (1,2,1,1)\rangle_{\mathbb{R}}$, find the distance from $(1,2,3,4)$ to W.

c) If $E \subset V$, then $(E^\perp)^\perp = E$ and $\dim(E) + \dim(E^\perp) = 4$.

d) Find the eigenvalues and eigenvectors of the orthogonal projection $T : V \to V$ onto E.

e) Is the above T diagonalizable?

Legendre Polynomials (Google it!)

Recall that $(f\,|\,g) = \int_{-1}^{1} f(x)g(x)dx$ is an inner product on $\mathcal{C}(I, \mathbb{R})$, where $I = [-1, 1]$. The polynomials $(L_0(t), L_1(t), \dots, L_n(t))$ resulting from the Gram-Schmidt orthonormalization of the standard basis $\mathcal{E} := (1, t, \dots, t^n)$ of $\mathsf{Pol}_n(\mathbb{R})$ are called the Legendre polynomials.

10) Answer / do the following:

a) Compute L_0, L_1, L_2, L_3.

b) Write $t^3 - t + 1$ as a linear combination of L_0, L_1, L_2, L_3.

c) Show that in general, one has $\deg(L_k) = k$.

d) What is the coefficient of t^k in $L_k(t)$?