Due: Fr, Apr 1, 2022

Math 314 / Problem Set 8 (two pages)

1) Prove **Cramer's Rule**: Let $A = (\mathcal{C}_1,\dots,\mathcal{C}_m) \in \mathrm{GL}_m(R)$, e.g. $A \in F^{m\times m}$ with $\det(A) \neq 0_F$. Then every system of linear equations $\mathcal{S} : A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution, given by:

$x_i = \det(A_i)/\det(A)$, $1 \leqslant i \leqslant m$, with $A_i \in R^{m\times m}$ obtained from A by replacing $\mathcal{C}_i$ by $\boldsymbol{b}$.

[**Hint**: Multiplying to the right by the adjoint $A^* = \big((-1)^{i+j}\Delta_{ji}\big)_{i,j}$ one gets $\det(A)\boldsymbol{x} = A^*\boldsymbol{b}$ (WHY).
Therefore, $\det(A)x_i = \sum_j (-1)^{i+j} b_j \Delta_{ji} = \det(A_i)$ $\forall\, i = 1,\dots,m$ (WHY), etc.]

• Study/read:

- Ch. 6 and 7 in *Linear Algebra* by Hoffman & Kunze.

(!) Make sure that you understand perfectly the definitions, examples, theorems, etc.

Diagonalization of matrices / linear transformations

2) Which of the matrices from HW # 7, Problems 7), 8) are diagonalizable? Find the diagonal form, in the case the matrix is diagonalizable.

3) Let $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ and $B = \begin{pmatrix} 5 & 2 \\ -3 & 0 \end{pmatrix} \in F^{2\times 2}$. Answer the following:

a) Find A^{2017} and e^A.

b) Find all $X \in F^{2\times 2}$ s.t. $X^2 = B$. Does the answer depend on F?

[**Hint:** Diagonalize first, etc.]

4) Prove/disprove/answer the following:

a) Let $A \in F^{5\times 5}$ have 3 distinct eigenvalues and a 3-dim eigenspace. Is A diagonalizable?

b) Give examples of 2×2 and 3×3 matrices over $\mathbb{C}$ which are not diagonalizable.

c) Suppose that $A \in F^{n\times n}$ satisfies: $A \neq \mathbf{0}_n$, $A^9 = \mathbf{0}_n$. Then A is not diagonalizable.

d) Suppose that $A \in \mathbb{Q}^{n\times n}$ satisfies: $A^2 = \mathbf{I}_n$. Then A is diagonalizable.
Does the same hold for $A \in \mathbb{C}^{n\times n}$, provided $A^k = \mathbf{I}_n$ for some $k > 0$?

5 Find the eigenvalues and the eigenspaces of the linear transformation $T : F^{2\times 2} \to F^{2\times 2}$ defined by $T(A) = A^\tau$. Can you give a generalization of this fact to $F^{n\times n}$?

6) Let $A \in F^{m\times m}$ with F algebraically closed. True of false (justify your answer):

a) If A has a unique eigenvalue, then A is diagonalizable if and only if A is diagonal.

b) Is the same true for "triangulate" instead of "diagonalize"?

c) If A is diagonalizable, then so is $p(A)$ for every $p(t) \in F[t]$.

d) Is the converse of assertion c) true?

[**Hint**: Recall/use the ideas form Problem 9 from HW 7, etc.]

7) Answer/prove/disprove the following:

a) If $A \in \mathbb{R}^{3\times 3}$ is not similar to upper triangular matrix, then A is diagonalizable over $\mathbb{C}$.

b) An upper triangular matrix $A \in F^{m\times m}$ which is diagonalizable is already diagonal.

Simultaneous diagonalization/triangulation

Recall that $\varphi_1, \ldots, \varphi_k \in \operatorname{End}_R(V)$ are said to commute with each other if $\varphi_i \varphi_j = \varphi_j \varphi_i$ for all $1 \leqslant i, j \leqslant k$. Further, $\varphi_1, \ldots, \varphi_k$ can be simultaneously diagonalized if there exists a basis $\mathcal{V}$ of V such that $[\varphi_i]_{\mathcal{V}}$ is diagonal for all $1 \leqslant i \leqslant k$. Define corresp. simultaneously triangulate. Similarly, $A_1, \ldots, A_k \in F^{m \times m}$ are simultaneously diagonalized if $\exists\, S \in \mathrm{GL}_m(F)$ such that SA_iS^{-1} are diagonal for $1 \leqslant i \leqslant k$. Define corresp. simultaneously triangulate.

8) Let $\varphi_1, \varphi_2 \in \operatorname{End}_R(V)$ be given. Prove/disprove/answer the following:

a) Let φ_1, φ_2 be diagonalizable. Then $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$ iff φ_1, φ_2 are simultaneously diagonalizable.

b) Is the same true if we replace "digonalize" by "upper triangulate"?

c) Are the above assertions true for any family of endomorphisms $\varphi_1, \ldots, \varphi_k$?

9) Answer/prove/disprove the following:

a) Given $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right),\ B = \left(\begin{smallmatrix} 1 & a \\ a & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$, find $S \in \mathrm{GL}_2(\mathbb{R})$ s.t. $S^{-1}AS,\ S^{-1}BS$ are diagonal.

b) Let $A_\alpha \in \mathbb{C}^{2\times 2}$, $\alpha \in I$ be a family of matrices such that $A_\alpha A_\beta = A_\beta A_\alpha\ \forall\, \alpha, \beta$. How many of the matrices A_α can be linearly independent over $\mathbb{C}$?

Solve as much as you can from the following:

- Problems **8, 9** at the end of section 6.6 of *Linear Algebra* by Hoffman & Kunze.
- Problems **4, 5, 8** at the end of section 7.3 of *Linear Algebra* by Hoffman & Kunze.