Due: Fr, March 25, 2022

Math 314 / Problem Set 7 (two pages)

• Study/read:

- *Determinants*: Ch. 5, *Linear Algebra* by Hoffman & Kunze & Ch.3. *LADW* by Treil.

- *Eigenvectors, etc.*: Ch. 6, *Linear Algebra* by Hoffman & Kunze & Ch.4. *LADW* by Treil.

(!) Make sure that you understand perfectly the definitions, examples, theorems, etc.

- *Polynomials*: Ch. 4, *Linear Algebra* by Hoffman & Kunze, especially Section 4.5.

Example: Let $p(t) = t^3 - 2t^2 + t$. Then one ca write $p(t) = t(t-1)^2$ (WHY), and we conclude: $t = 0$ is root of multiplicity $m = 1$; $t = 1$ is a root of multiplicity $m = 2$, and $t = a$ is a root of multiplicity $m = 0$ for all $a \in F$, $a \neq 0_F, 1_F$.

- Find the roots of $p(t)$ and their multiplicities in each of the fields $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_2, \mathbb{F}_3$:
  $p(t) = t^7 - 2t^5 + t^3$; $p(t) = t^9 - 8t^5 + 16t$; $p(t) = t^6 - 1$; $p(t) = 4t^4 + 1$; $p(t) = x^n - 1$.

Determinants

1) For $A = (a_{ij})_{i,j} \in F^{5\times 5}$, which of the following monomials are summands in $\det(A)$?

$$-a_{53}a_{11}a_{23}a_{45}a_{51};\quad -a_{11}a_{22}a_{33}a_{45}a_{54};\quad a_{12}a_{23}a_{34}a_{45}a_{54};\quad a_{12}a_{23}a_{34}a_{45}a_{51}.$$

2) Let $A = (a_{ij})_{i,j}$ be an $m \times m$ matrix whose coefficients are *polynomials* $a_{ij} = a_{ij}(t)$ from $\mathsf{Pol}_n(F)$. True of false (justify your answer):

a) $\det(A)$ is a polynomial, say $\Delta(t) := \det(A)$, of degree $\leq mn$.
b) If each $a_{ij}(t)$ has $t = a$ as a root, then $t = a$ is a root of $\Delta(t)$ of multiplicity $\geq m$.
c) If none of the entries $a_{ij}(t)$ is constant, then $\Delta(t)$ is not constant.
d) If none of the entries $a_{ij}(t)$ has a root in F, then $\Delta(t)$ has no roots in F.

3) Let $A = \big(f_{ij}(x)\big)_{i,j}$ be an $m \times m$ matrix having as entries differentiable functions $f_{ij}(x)$ on some interval $I = (a, b)$. Answer the following:

a) For $m = 2$ show that $f(x) := \det(A)$ is a differentiable function on I, and find $f'(x)$ in terms of $f_{ij}(x)$ and their derivatives $f'_{ij}(x)$.
b) Try to generalize the above result general $m \times m$ matrices.

[**Hint:** Show that if $A = (\mathcal{C}_1, \dots, \mathcal{C}_m)$, then $f'(x) = |\mathcal{C}'_1, \mathcal{C}_2, \dots, \mathcal{C}_m| + |\mathcal{C}_1, \mathcal{C}'_2, \dots, \mathcal{C}_m| + \dots + |\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}'_m|$, etc.]

4) Prove/disprove/answer the following:

a) Suppose that $A \in \mathbb{Z}^{m\times m}$. Then $A^{-1} \in \mathbb{Z}^{m\times m}$ if and only if $\det(A) = \pm 1$.
b) For $m = 2$, give examples with $A, A^{-1} \in \mathbb{Z}^{2\times 2}$ whose coefficients a_{ij} satisfy $|a_{ij}| > 100$.
c) Is there some $A \in \mathbb{C}^{7\times 7}$ such that $\lambda \boldsymbol{I}_7 - A$ is not invertible for eight distinct values $\lambda \in \mathbb{C}$? Is the same true over every field F?
d) What is the corresponding assertion for $A \in \mathbb{C}^{m\times m}$?

Minors and rank of a matrix

Let $A = (a_{ij})_{i,j} \in F^{m\times n}$ and $k \le m, n$ be a positive integer. Recall that the rank k minors of A are the $k\times k$ determinants $\Delta_{i_1,\ldots,i_k;\, j_1,\ldots,j_k}$ whose entries are the elements a_{ij} at the intersection of the k rows $\boldsymbol{\mathcal{R}}_{i_1}, \ldots, \boldsymbol{\mathcal{R}}_{i_k}$ with the columns $\boldsymbol{\mathcal{C}}_{j_1}, \ldots, \boldsymbol{\mathcal{C}}_{j_k}$ of A. Thus the rank 1 minors of A are simply the coefficients a_{ij} of A; the rank 2 minors are all the determinants of the form $\begin{vmatrix} a_{ij} & a_{ij'} \\ a_{i'j} & a_{i'j'} \end{vmatrix}$ with $1 \le i < i' \le m$, $1 \le j < j' \le n$ (WHY), etc.

5) Prove the following:

a) Suppose that $\operatorname{rank}(A) = 2$. Then all the rank 3 minors A have value 0_F.

b) Show that the following conditions are equivalent:

  i) Any k rows $\boldsymbol{\mathcal{R}}_{i_1}, \ldots, \boldsymbol{\mathcal{R}}_{i_k}$ or columns $\boldsymbol{\mathcal{C}}_{j_1}, \ldots, \boldsymbol{\mathcal{C}}_{j_k}$ are linearly dependent.

  ii) All the k-minors are zero.

c) Conclude that $\operatorname{rank}(A) = k$ iff there exists a rank k minor of A which is non-zero.

Eigenvectors/Eigenvalues

6) True of false (justify if not obvious):

a) $0_V \in V$ is an eigenvector for all $T \in \operatorname{End}(V)$.

b) If $\dim(V) = n$, then every $T \in \operatorname{End}(V)$ has n distinct eigenvalues.

b) If $A \in F^{n\times n}$ has *one* eigenvector, it has *infinitely many* eigenvectors.

c) There exists a matrix $A \in \mathbb{R}^{n\times n}$ with no eigenvalues in $\mathbb{R}$.

d) There exists a matrix $A \in \mathbb{C}^{n\times n}$ with no eigenvalues in $\mathbb{C}$.

e) Similar matrices always have the same eigenvalues.

f) Similar matrices always have the same eigenvectors.

g) The sum of two eigenvectors of a matrix A is always an eigenvector.

h) If v_1, v_2 are eigenvectors to the same eigenvalue, then so are $v_1 \pm v_2$.

7) Find the characteristic polynomials, eigenvalues and eigenvectors over $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_2, \mathbb{F}_3$ of:

$$\text{a)}\ \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix} \qquad \text{b)}\ \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \qquad \text{c)}\ \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \qquad \text{d)}\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix}$$

8) Same question for matrices below over the fields $\mathbb{R}, \mathbb{C}$:

$$\text{a)}\ \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & e & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \text{b)}\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad \text{b)}\ \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \pi \end{pmatrix}$$

9) Let $A \in F^{n\times n}$ over a field F be given. Prove/disprove the following:

a) If A is *nilpotent,* i.e., $\exists\, k > 0$ s.t. $A^k = \mathbf{0}_{n\times n}$, then $\lambda = 0_F$ is the unique eigenvalue of A.

j) If A is a *projection* matrix, i.e, $A^2 = A$, then A can have only $0_F, 1_F$ as eigenvalues.

c) If λ is an eigenvalue for A, then $\mu := \lambda^2 - 3\lambda$ is an eigenvalue for $B := A^2 - 3A$.

d) If λ is an eigenvalue for A, then λ^k is eigenvalue for $B := A^k$.

e) What is the generalization of c) and d) above?