Due: Fr, March 18, 2022

Math 314 / Problem Set 6 (two pages)

- **Study/read:** *Determinants*

typed Notes; Ch 3 from *LADW* by Treil; Ch 5 from *Linear Algebra* by Hoffman & Kunze.

(!) Make sure that you understand perfectly the definitions, examples, theorems, etc.

1) Permutations. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 3 & 2 \end{pmatrix} \in S_5$ be given. Answer the following questions:

a) Compute $\epsilon(\sigma)$, σ^2, σ^{-1} and find N minimal such that $\sigma^N = \text{id}$.
b) Define $A \in R^{5\times 5}$ by $a_{ij} = \delta_{\sigma(i)j}$. Then $A \in \text{GL}_5(R)$, and $A^N = \boldsymbol{I}_5$.
c) The number N_0 of even permutations in S_n equals the number N_1 of odd permutations. Is $N_0 = N_1$ alway an even number? **Justify your answer!**

2) True of false (justify if not obvious):

a) If the third column of a matrix is the difference of the first and second columns of that matrix then the matrix is not invertible.
b) If $\det(AB) \neq 0$, then $\det(A) \neq 0$ and $\det(B) \neq 0$.
c) Let $F = \mathbb{Q}$ be the field of rational numbers. Then there is no $A \in F^{n\times n}$ such that $\det(AA^\tau) = 2$.
d) If $A \in F^{n\times n}$ is an $n \times n$ matrix, then $\text{rank}(A) \neq n$ iff $\det(A) = 0$.
e) The vector space of continuous functions $\mathcal{C}(\mathbb{R}, \mathbb{R})$ has a basis consisting of analytic functions.

3) True of false (justify if not obvious):

a) For $A \in \mathbb{R}^{4\times 4}$ one has: The system of equations $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{b} \in \mathbb{R}^{4\times 1}$ has a unique solution iff $\det(A) \neq 0$.
b) A diagonal matrix A has $\det(A) = 0$ iff some element on the diagonal of A is zero.
c) Every symmetric matrix A has $\det(A) \neq 0$.
d) Every anti-symmetric matrix $A \in F^{9\times 9}$ has $\det(A) = 0_F$. (Dependence on F?)
e) Let $A, E_1, \dots, E_r \in F^{n\times n}$ with the $E_1, \dots, E_r$ elementary matrices. Then one has that $\det(A) = 1_F$ iff $\det(AE_1 \dots E_r) = 1_F$, and $\det(A) = 0_F$ iff $\det(AE_1 \dots E_r) = 0_F$.

4) Answer the following:

a) If $A \in F^{n\times n}$, how are the determinants $\det(A)$ and $\det(5A)$ related?
b) How are related the following determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \begin{vmatrix} -3a_1 & 2a_2 & 5a_3 \\ -3b_1 & 2b_2 & 5b_3 \\ -3c_1 & 2c_2 & 5c_3 \end{vmatrix} \qquad \begin{vmatrix} 3a_1 & 4a_2 + 5a_1 & -2a_3 \\ 3b_1 & 4b_2 + 5b_1 & -2b_3 \\ 3c_1 & 4c_2 + 5c_1 & -2c_3 \end{vmatrix}$$

c) Compute the following determinants using row/column operations:

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & -2 & 3 \\ 3 & 1 & 1 & -2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{vmatrix}$$

5) Let $A \in F^{n\times n}$ be given. Answer the following:

a) If A is skew-symmetric and n is odd, then $\det(A) = 0_F$. Is this true for even n?

b) A is called nilpotent, if $A^k = \mathbf{0}_n$ for some positive integer number $k > 0$. Show that for a nilpotent matrix A one has $\det(A) = 0_F$. Is the converse true?

c) B and A are called similar, if $B = S^{-1}AS$ for some invertible matrix $S \in F^{n\times n}$. What is the relation between $\det(A)$ and $\det(B)$, provided A and B are similar?

d) A is called orthogonal, if $AA^\tau = I_n$. Prove that A orthogonal implies $\det(A) = \pm 1$.

6) Let $x_1, \ldots, x_n$ be elements of F. The *Vandermonde determinant* defined by $x_1, \ldots, x_n$ is:

$$V(x_1, \ldots, x_n) := \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}$$

a) Compute $V(x, y)$ and $V(x, y, z)$, and show that they vanish iff $x = y$, respectively $x = y$ or $x = z$ or $y = z$.

b) Google the term "Vandermonde determinant" and learn/prove the general formula for $V(x_1, \ldots, x_n)$.

7) Let $A \in F^{n\times n}$, $C \in F^{m\times m}$, $B \in F^{m\times n}$, $\tilde{B} \in F^{n\times m}$, $O := \mathbf{0}_{m\times n}$, $\tilde{O} := \mathbf{0}_{n\times m}$.

a) Compute the determinants of the "block matrices" below:

$$\begin{pmatrix} \boldsymbol{I}_m & B \\ \tilde{O} & A \end{pmatrix} \qquad \begin{pmatrix} A & \tilde{O} \\ B & \boldsymbol{I}_m \end{pmatrix} \qquad \begin{pmatrix} \boldsymbol{I}_n & \tilde{B} \\ O & C \end{pmatrix} \qquad \begin{pmatrix} C & O \\ \tilde{B} & \boldsymbol{I}_n \end{pmatrix}$$

b) For $m = n$, can one prove that $\det(AC) = \det(A)\det(C)$ using the facts above?

[**Hint:** Maybe involving that $\begin{pmatrix} C & B \\ \tilde{O} & A \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_m & B \\ \tilde{O} & A \end{pmatrix} \begin{pmatrix} C & O \\ \tilde{O} & \boldsymbol{I}_m \end{pmatrix}$, etc.]

8) Solve problems **7.1, 7.2, 7.3, 7.4** from Ch. 3 of *Linear Algebra Done Wrong* by Treil.