Due: Wed, March 2, 2022

Math 314 / Problem Set 5 (two pages)

• Study/read:

Ch 2 & 3 from *Linear Algebra* by Hoffman & Kunze or Ch 1 & 2 from *LADW* by Treil.

Elementary transformations / Equivalent systems of linear equations

- For $A = (\boldsymbol{\mathcal{R}}_i)_i = (\boldsymbol{c}_j)_j \in R^{m\times n}$ recall the row/column modules $\boldsymbol{\mathcal{R}}_A = \langle(\boldsymbol{\mathcal{R}}_i)_i\rangle_R$, respectively $\boldsymbol{\mathcal{C}}_A = \langle(\boldsymbol{c}_j)_j\rangle_R$ of A, and their ranks $\mathrm{rk}(\mathcal{R}_A)$, $\mathrm{rk}(\mathcal{C}_A)$ if defined, e.g. if $R = F$ is a field. Make sure that you check/know the details of the proofs:
  - $\boldsymbol{\mathcal{R}}_{EA} \subset \boldsymbol{\mathcal{R}}_A$, and if $E \in \mathrm{GL}_m(R)$ then $\boldsymbol{\mathcal{R}}_{EA} = \boldsymbol{\mathcal{R}}_A$.
  - $\boldsymbol{\mathcal{C}}_{AE} \subset \boldsymbol{\mathcal{C}}_A$, and if $E' \in \mathrm{GL}_n(R)$ then $\boldsymbol{\mathcal{C}}_{AE} = \boldsymbol{\mathcal{C}}_A$.

  (∗) Hence $\boldsymbol{\mathcal{R}}_A$ is invariant under invertible elementary row operations, and so is $\mathrm{rk}(\boldsymbol{\mathcal{R}}_A)$, if defined. Correspondingly, the same holds for $\boldsymbol{\mathcal{C}}_A$, and column transformations.
  - If A is in reduced row form and R is a domain, $\boldsymbol{\mathcal{R}}_A$ is free. What is the rank?
  - If $R = F$ is a field, the row/column reduced echelon form is unique.

1) Let F be a field, and $A \in F^{m\times n}$. Answer/prove/disprove:

a) Give an estimate of the number of multiplication/divisions necessary to bring A into its row reduced echelon form.

b) There are elementary matrices $E_1, \dots, E_s \in F^{m\times m}$ and $E'_1, \dots, E'_{s'} \in F^{n\times n}$ s.t. $E_s \dots E_1 A E'_1 \dots E'_{s'} = \begin{pmatrix} \boldsymbol{I}_k & \mathbf{0} \\ \mathbf{0}'' & \mathbf{0}' \end{pmatrix} =: \mathcal{I}_A$ with $\mathbf{0}, \mathbf{0}', \mathbf{0}''$ are zero-matrices (maybe empty).

c)* $\mathcal{I}_A$ is the row & column reduced echelon form of A. Prove that it is unique.

d)* What would be the "column & row reduced echelon form" of A?

2) Let $A, B \in R^{m\times n}$ be in *row reduced* ***echelon*** *form.* Prove/answer the following:

a) Let $f_A, f_B : R^{n\times 1} \to R^{m\times 1}$ be the canonical R homomorphisms defined by A, respectively B. Show that $\ker(f_A) = \ker(f_B)$ iff $A = B$.

b) Deduce from this: *the row reduced echelon form over R is unique*, if it exists.

- Recall that two systems of linear equations $\mathcal{S}$: $A\boldsymbol{x} = \boldsymbol{b}$, $\mathcal{S}'$: $A'\boldsymbol{x} = \boldsymbol{b}'$ with $A, A' \in R^{m\times n}$, $\boldsymbol{b}, \boldsymbol{b}' \in R^{m\times 1}$ are called equivalent, if they have equal sets of solutions $Sol(\mathcal{S}') = Sol(\mathcal{S})$. And recall the homogeneous system $\mathcal{S}_0 : A\boldsymbol{x} = \mathbf{0}_m$ attached to $\mathcal{S}$. Make sure that you know/check all the details in the proofs of the following:
  - $Sol(\mathcal{S}) = x_0 + Sol(\mathcal{S}_0)$ for every $x_0 \in Sol(\mathcal{S})$.
  - If $A' = EA$ and $\boldsymbol{b}' = E\boldsymbol{b}$ for some $E \in \mathrm{GL}_m(R)$, then $\mathcal{S}$, $\mathcal{S}'$ are equivalent.
  - If A is in row reduced echelon form, $\mathcal{S}$: $A\boldsymbol{x} = \mathbf{0}_m$ has the *standard basis* given in class.

3) Let $\mathcal{S}$: $A\boldsymbol{x} = \boldsymbol{b}$ and $\mathcal{S}'$: $A'\boldsymbol{x} = \boldsymbol{b}'$ be equivalent systems of equations. Answer:

a) If $A' = EA$ and $\boldsymbol{b}' = E\boldsymbol{b}$ for some $E \in \mathrm{GL}_m(R)$, is the matrix E unique ?

d) Is there always some $E \in \mathrm{GL}_m(R)$ such that $A' = EA$, $\boldsymbol{b}' = E\boldsymbol{b}$?

4) Let $f : W \to V$ be a morphism of F-vector spaces with bases $\mathcal{B} = (\beta_1, \dots, \beta_n)$, resp. $\mathcal{A} = (\alpha_1, \dots, \alpha_m)$, and $[\ ]_{\mathcal{A}} : V \to F^{m\times 1}$, $[\ ]_{\mathcal{B}} : W \to F^{n\times 1}$ be the coordinate isomorphisms. Set $A := [f]_{\mathcal{AB}}$, and consider the system of linear equations $\mathcal{S}_0$: $A\boldsymbol{x} = \mathbf{0}_m$. Prove:

a) $[\ ]_{\mathcal{A}} : \mathrm{Im}(f) \to \boldsymbol{\mathcal{C}}_A$ and $[\ ]_{\mathcal{B}} : \mathrm{Ker}(f) \to \mathit{Sol}(\mathcal{S}_0)$ are isomorphisms of F-vector space.

b) The **rank-nullity formula** holds: $n = \dim\big(\mathrm{Ker}(f)\big) + \dim\big(\mathrm{Im}(f)\big)$.

[**Hint:** To b): For $(\alpha_i)_i$ basis of $\mathrm{Ker}(f)$, $\big(f(\beta_j)\big)_j$ basis of $\mathrm{Im}(f)$, have: $\big((\alpha_i)_i, (\beta_j)_j\big)$ is a basis of W (WHY), etc.]

5) For $a, b, c, b_1, \dots, b_m \in R$ arbitrary, discuss/describe $\mathit{Sol}(\mathcal{S}), \mathit{Sol}(\mathcal{S}_0)$ for $\mathcal{S}\colon A\boldsymbol{x} = \boldsymbol{b}$ below.

a) $\begin{pmatrix} a & 1 & 2 \\ 1 & 1 & b \\ 2 & b & c \end{pmatrix}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ c \end{pmatrix}$ b) $\begin{pmatrix} 0 & 3 & 1 & 2 & 1 \\ 2 & 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & -1 \end{pmatrix}\boldsymbol{x} = \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}$ c) $\begin{pmatrix} 1 & 2 & \dots & n \\ 2 & 3 & \dots & n+1 \\ \vdots & \vdots & & \\ m & m+1 & \dots & m+n-1 \end{pmatrix}\boldsymbol{x} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

About S_n and $\mathcal{I}$, $\mathcal{I}_{\leqslant}$, $\mathcal{I}_{<}$

• Recall that for $m, n \geq 0$ one defines $\mathcal{I} := \{(i_1, \dots, i_n) \,|\, 1 \leqslant i_1, \dots, i_n \leqslant m\}$ as well as its subsets $\mathcal{I}_{\leqslant} := \{(i_1, \dots, i_n) \,|\, i_1 \leqslant \dots \leqslant i_n\}$ and $\mathcal{I}_{<} := \{(i_1, \dots, i_n) \,|\, i_1 < \dots < i_n\}$, and the action of S_n on $\mathcal{I}$ by $\sigma : \mathcal{I} \to \mathcal{I}$, $\underline{\boldsymbol{\imath}} := (i_1, \dots, i_n) \mapsto (i_{\overline{\sigma}(1)}, \dots, i_{\overline{\sigma}(n)}) =: \sigma(\underline{\boldsymbol{\imath}})$.

6) Prove/disprove:

a) The map $\sigma : \mathcal{I} \to \mathcal{I}$ is a bijection for all $\sigma \in S_n$.

b) For every $\underline{\boldsymbol{\imath}}' \in \mathcal{I}$, there exists a unique $\underline{\boldsymbol{\imath}} \in \mathcal{I}_{\leqslant}$ such that $\underline{\boldsymbol{\imath}}' = \sigma(\underline{\boldsymbol{\imath}})$ for some $\sigma \in S_n$.

c) If $\underline{\boldsymbol{\imath}}' = (i'_1, \dots, i'_n) \in \mathcal{I}$ has all $i'_1, \dots, i'_n$ distinct, then there exit unique $\underline{\boldsymbol{\imath}} \in \mathcal{I}_{\leqslant}$ and $\sigma \in S_n$ such that $\underline{\boldsymbol{\imath}}' = \sigma(\underline{\boldsymbol{\imath}})$ as above; and in particular, $\underline{\boldsymbol{\imath}} \in \mathcal{I}_{<}$.

7) Answer the following:

a) Find the cardinality of the sets $\mathcal{I}, \mathcal{I}_{\leqslant}, \mathcal{I}_{<}$ (Google it ?).

b) In the notation from Problem 6 b) above, find all σ and $\underline{\boldsymbol{\imath}}$, in the following cases:

- $m = 7$, $n = 5$, and $\underline{\boldsymbol{\imath}}' = (4, 7, 3, 4, 1)$, $\underline{\boldsymbol{\imath}}' = (1, 7, 1, 2, 1)$.
- $m = 4$, $n = 7$, and $\underline{\boldsymbol{\imath}}' = (4, 4, 3, 4, 1, 2, 2)$, $\underline{\boldsymbol{\imath}}' = (4, 3, 2, 1, 2, 1, 1)$.

• Let M, N be R-modules, $n \geq 1$, and recall the notations from the class:

$$\mathcal{L}^n_{\mathrm{sym}}(M, N), \mathcal{L}^n_{\mathrm{alt}}(M, N) \subset \mathcal{L}^n(M, N) \subset \mathrm{Maps}(M^n, N).$$

Make sure that you know/check all the details in the proofs of the assertions from the class:

- $\mathcal{L}^n_{\mathrm{sym}}(M, N), \mathcal{L}^n_{\mathrm{alt}}(M, N) \subset \mathcal{L}^n(M, N) \subset \mathrm{Maps}(M^n, N)$ are R-submodules.
- What is $\mathcal{L}^n_{\mathrm{sym}}(M, N) \cap \mathcal{L}^n_{\mathrm{alt}}(M, N)$ inside $\mathcal{L}^n(M, N)$?
- $f \in \mathcal{L}^n_{\mathrm{sym}}(M, N)$ iff $f(\dots, x_i, \dots, x_j, \dots) = f(\dots, x_j, \dots, x_i, \dots)$ $\forall$ $(x_1, \dots, x_n) \in M^n$, $i < j$.
- $f \in \mathcal{L}^n_{\mathrm{alt}}(M, N)$ iff $f(x_1, \dots, x_n) = 0$ $\forall$ $(x_1, \dots, x_n) \in M^n$ if $x_i = x_j$ for some $i < j$.

• Suppose that $2x = 0_N \Rightarrow x = 0_N$ for all $x \in N$. Check the details in the proof of the following characterization of $f \in \mathcal{L}^n(M, N)$ being alternating. TFAE:

i) f is alternating, i.e., $f \in \mathcal{L}^n_{\mathrm{alt}}(M, N)$.

ii) $\forall$ $(x_1, \dots, x_n) \in M^n$ and $\forall$ $i < j$ one has $f(\dots, x_i, \dots, x_j, \dots) = -f(\dots, x_j, \dots, x_i, \dots)$.

[**Hint:** Evaluate $f(\dots, x_i + x_j, \dots, x_i + x_j, \dots)$, and use that S_n is generated by transpositions, etc...]