Due: We, Feb 23, 2022

Math 314/514 Problem Set 4 (two pages)

Study/read:

Ch 2 & 3 from *Linear Algebra* by Hoffman & Kunze or Ch 1 & 2 from *LADW* by Treil.

Review/complete all the details of the proofs of the following assertions from class (although some assertions hold more general, we assume that R is commutative, $1_R \neq 0_R$).

- Matrices:
  - Multiplication of matrices, when defined, is associative.
  - For all $A \in R^{m\times n}$, one has $\boldsymbol{I}_m A = A = A\boldsymbol{I}_n$, where $\boldsymbol{I}_m, \boldsymbol{I}_n$ are the identity matrices.
  - $\varphi : R \to R^{m\times m},\ a \mapsto a\boldsymbol{I}_m$ is a ring morphism s.t. $\varphi(a)A = A\varphi(a)$ for all a, A.

- Coordinate vectors/maps: M is an R-module, $\mathcal{E} = (e_1, \dots, e_m)$ standard basis of $R^{m\times 1}$.
  - Let $\mathcal{A} = (\alpha_1, \dots, \alpha_m)$ be a basis of M. The coordinate map $[\ ]_{\mathcal{A}} : M \to R^{m\times 1}$ defined by $x \mapsto [x]_{\mathcal{A}}$ is an isomorphism of R-modules which maps $\mathcal{A}$ onto $\mathcal{E}$.
  - Conversely, if $\imath : M \to R^{m\times 1}$ is an isomorphism of R-modules, and $\mathcal{A} := \imath^{-1}(\mathcal{E})$, then $\mathcal{A}$ is a basis of M such that $[x]_{\mathcal{A}} = \imath(x)$ for all $x \in M$.

- Change of basis: Let $\mathcal{A} = (\alpha_1,..., \alpha_m)$, $\mathcal{A}' = (\alpha'_1,..., \alpha'_m)$ be generators of an R-module M.
  - Suppose that $\mathcal{A}, \mathcal{A}'$ are bases of M. Then there exists a *unique* matrix $S_{\mathcal{A}'\mathcal{A}} \in \mathrm{GL}_m(R)$ s.t. $\mathcal{A} = \mathcal{A}'S_{\mathcal{A}'\mathcal{A}}$. Further, $[x]_{\mathcal{A}'} = S_{\mathcal{A}'\mathcal{A}}[x]_{\mathcal{A}}$.
  - Conversely, suppose that $\mathcal{A} = \mathcal{A}'S$ for some $S \in \mathrm{GL}_m(R)$. Then $\mathcal{A}$ is a basis of M iff $\mathcal{A}'$ is a basis of M. If so, then $S = S_{\mathcal{A}'\mathcal{A}}$ is the matrix from the previous item a).

- Morphisms & Matrices, Base change formulas: Let $\mathcal{A} = (\alpha_1, \dots, \alpha_m)$, $\mathcal{A}' = (\alpha'_1, \dots, \alpha'_m)$ and $\mathcal{B} = (\beta_1, \dots, \beta_n)$, $\mathcal{B}' = (\beta'_1, \dots, \beta'_n)$ be bases of the the R-modules M, respectively N.

Recall/review/check all the details in the proofs of the following basic facts from the class:

- $\Psi_{\mathcal{A}\mathcal{B}} : \mathrm{Hom}_R(N, M) \to R^{m\times n},\ f \mapsto [f]_{\mathcal{A}\mathcal{B}}$ is an isomorphism of R-modules.
- $\Psi_{\mathcal{A}} : \mathrm{End}_R(M) \to R^{m\times m},\ f \mapsto [f]_{\mathcal{A}} := [f]_{\mathcal{A}\mathcal{A}}$ is an isomorphism of R-algebras.
- One has: $[f(y)]_{\mathcal{A}} = [f]_{\mathcal{A}\mathcal{B}}[y]_{\mathcal{B}}$ for all $y \in N$.
- What are the columns of $S_{\mathcal{A}'\mathcal{A}}$, respectively $S_{\mathcal{A}\mathcal{A}'}$?
- For $f \in \mathrm{Hom}_R(N, M)$, $g \in \mathrm{End}_R(M)$, one has the base change formulas:

$$[f]_{\mathcal{A}'\mathcal{B}'} = S_{\mathcal{A}'\mathcal{A}}[f]_{\mathcal{A}\mathcal{B}}S_{\mathcal{B}\mathcal{B}'}, \quad [g]_{\mathcal{A}'} = S_{\mathcal{A}'\mathcal{A}}[g]_{\mathcal{A}}S_{\mathcal{A}\mathcal{A}'} = S_{\mathcal{A}'\mathcal{A}}[g]_{\mathcal{A}}S^{-1}_{\mathcal{A}'\mathcal{A}}.$$

Basics about matrices

1) Are the assertions from Change of basis above true if $\mathcal{A}$, $\mathcal{A}'$ are not necessarily bases? Is it true that for $A \in R^{m\times n}$ one has: $A = \mathbf{0}_{m\times n}$ iff $Ab = \mathbf{0}_n$ for all $b \in R^{n\times 1}$?

2) Recall that $A = (a_{ij})_{i,j} \in R^{n\times n}$ is called upper diagonal if $a_{ij} = 0$ for $i > j$. Prove/disprove: *An upper diagonal matrix $A = (a_{ij})_{i,j} \in R^{n\times n}$ is invertible iff $a_{ii} \in R^\times$ for all $i = 1, \dots, n$.*

3) For $m < n$ let A be an $m \times n$ matrix and B be an $n \times m$ matrix.

a) Give examples to show that AB can be invertible, but BA is not invertible.

b)* Are there examples in which BA is invertible?

4) Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Answer the following:

a) Suppose that $m = 2 = n$. Show that if $AB = \boldsymbol{I}_2$, then $BA = \boldsymbol{I}_2$. Is it true that $m = n$ and $AB = \boldsymbol{I}_m$ always imply $BA = \boldsymbol{I}_m$?

b) Suppose that $m, n \leq 2$. Show that if $AB = \boldsymbol{I}_m$ and $BA = \boldsymbol{I}_n$, then $m = n$. Is the same true correspondingly for all m, n?

Review/complete all the details of the proofs of the following assertions from class (although some assertions hold more general, we assume that R is commutative, $1_R \neq 0_R$).

- Elementary matrices/matrix operations: Check all the details in the proof of the following assertion from the class:

- If $A \in R^{m\times n}$ or $A \in M^{m\times n}$ has rows $\boldsymbol{\mathcal{R}}_1, \dots, \boldsymbol{\mathcal{R}}_m$, then $\boldsymbol{e}_{kl}A$ has rows $\delta_{1k}\boldsymbol{\mathcal{R}}_l, \dots, \delta_{mk}\boldsymbol{\mathcal{R}}_l$.
- If A has columns $\boldsymbol{\mathcal{C}}_1, \dots, \boldsymbol{\mathcal{C}}_n$, then $A\boldsymbol{e}_{kl}$ has columns $\delta_{1l}\boldsymbol{\mathcal{C}}_k, \dots, \delta_{nl}\boldsymbol{\mathcal{C}}_m$.
- $E_k(a)$ is invertible iff $a \in R^\times$. Second, $E(k, l)$ and $E_{kl}(a)$ $(k \neq l)$ are always invertible.
- EA is the result of the corresponding row operation on $A \in R^{m\times n}$ given in class.
- Correspondingly the same for the column operations.

5) Let $E_1(a)$, $E(1, 2)$ and $E_{12}(a)$ be the elementary matrices as defined in class. Prove/disprove:

a) $E_1(a)$ commutes with $E_k(b)$, $E(k, l)$, and $E_{kl}(b)$, provided $k, l \neq 1$. Is the converse true? What is the general commuting assertion for $E_i(a)$?

b) $E(1, 2)$ commutes with $E(k, l)$, respectively $E_{kl}(b)$, provided $k, l \neq 1, 2$. Is the converse true? What is the general commuting assertion for $E(i, j)$?

c) $E_{12}(a)$ commutes with $E_{kl}(b)$, provided $k, l \neq 1, 2$. Is the converse true? What is the general commuting assertion for $E_{ij}(a)$?

6) Using elementary row transformations, compute the row/column reduced (echelon) form of the matrices below; if invertible, find their inverses. (Consider $\overline{0}, \overline{1}$ in $\mathbb{F}_2$ or $\mathbb{F}_3$ separately.)

a) $\begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 1 \\ 2 & 3 & i+1 \\ 4 & 9 & 2i \end{pmatrix}$ c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ d) $\begin{pmatrix} 0 & 3 & 1 & 2 & 1 \\ 2 & 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & -1 \end{pmatrix}$ e) $\begin{pmatrix} \overline{0} & \overline{1} & \overline{1} & \overline{0} \\ \overline{1} & \overline{1} & \overline{1} & \overline{0} \\ \overline{1} & \overline{0} & \overline{1} & \overline{0} \\ \overline{1} & \overline{0} & \overline{1} & \overline{1} \end{pmatrix}$

7) The famous Hilbert matrices are $n \times n$ *symmetric* matrices from $\mathbb{Q}^{n\times n}$ of the form:

$$H(n) = \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

a) Show that $H(2)$, $H(3)$, $H(4)$ are invertible, and the inverses have integer coefficients.

b) Try to prove that $H(n)$ are invertible, and the inverse matrix has integer coefficients.

($*$) *Google the term Hilbert matrix and learn more about it.*