Due: Mo, Feb 14, 2022

Math 314 / Problem Set 3 (two pages)

- **Study/read:**

Ch 2 & 3 from *Linear Algebra* by Hoffman & Kunze or Ch 1 & 2 from *LADW* by Treil.

1) Indicate which of the following sets (with the natural addition and scalar multiplication) are $\mathbb{R}$-vector spaces, and if so, give the zero vector. Justify your answer!

a) The set $\mathcal{C}(I,\mathbb{R})$ of all continuous functions on the interval $I := (0,1)$.

b) The set $\mathcal{C}(I,\mathbb{R})_{\geqslant 0}$ of all non-negative continuous functions on $I := (0,1)$.

c) The set $\mathcal{C}^1(I,\mathbb{R})$ of all differentiable functions with continuos derivative on $I := (0,1)$.

d) The set $V \subset \mathsf{Pol}_n(\mathbb{R})$ of all polynomials $p(t) \in \mathbb{R}[t]$ of degree $\leqslant n$ satisfying $p(-1) = 0$.

e) The set $V_n \subset \mathsf{Pol}_n(\mathbb{R})$ all polynomials $p(t) \in \mathbb{R}[t]$ of degree exactly n.

f) The set of all symmetric 3×3 matrices $A \in \mathbb{R}^{3\times 3}$ i.e. the set of matrices $A = (a_{ij})_{i,j}$ such that $a_{ij} = a_{ji}$ for all $1 \leqslant i, j \leqslant 3$.

g) The set of all the matrices 2×3 matrices X satisfying $\begin{pmatrix} \frac{7}{2} & \frac{2}{9} \\ \frac{1}{1} & \frac{1}{10} \end{pmatrix} X \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} = X$.

h) The set of all real valued functions $f \in \mathcal{F}(X,\mathbb{R})$ such that $|f(X)| \leqslant 3$.

2) Let $\mathsf{Pol}_n \subset R[t]$ be the R-submodule of the polynomials of degree $\leqslant n$. Prove/disprove/answer the following (**Note:** The answer might depend on the ring R):

a) $\mathcal{R} := \{p(t) \in \mathsf{Pol}_n \,|\, p(t) \text{ has a root in } R\}$ is an R-submodule of Pol_n.

b) The subset $\mathcal{Q} \subset \mathsf{Pol}_n$ of all the squares of polynomials is a submodule of Pol_n.

[**Hint** to b): If $2\cdot 1_R \neq 0_R$, the answer is NO (WHY). If $R = \mathbb{F}_2$, prove by induction on $d := \deg\big(f(t)\big)$ that $f(t)$ is even iff $f(t) = p(t)^2$ for some polynomial $p(t)$, etc. Examples: $t^2 + \overline{1} = (t + \overline{1})^2$, $t^8 + t^6 + t^2 = (t^4 + t^3 + t)^2$, etc.]

3) Let V be an F-vector space. Prove/disprove:

a) If $V_1, V_2 \subseteq V$ are subspaces, then the union $V_1 \cup V_2$ is a subspace.

b)* If $V_1, V_2 \subseteq V$ are subspaces such that $V_1 \cup V_2$ is a subspace, then $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

4) Let $\mathsf{Pol}_2(\mathbb{R}) = \{p \in \mathbb{R}[t] \,\big|\, \deg(p) \leqslant 2\} \subset \mathbb{R}[t]$, and for $p(t) \in \mathsf{Pol}_2(\mathbb{R})$ let $p^{(i)}(t)$ with $i \geqslant 0$ denote its derivatives. Let $a_0, a_1, a_2 \in \mathbb{R}$ be fixed. Prove/disprove:

a) $V_i := \{p(t) \in \mathsf{Pol}_2(\mathbb{R}) \mid p(0) = \cdots = p^{(i)}(0) = 0\}$ are subspaces of $\mathsf{Pol}_2(\mathbb{R})$ for all $i \geq 0$.

b) Give concrete descriptions of $V_0 \cap V_1$, $V_1 \cap V_2$, and $V_0 \cap V_1 \cap V_2$ and $V_0 + V_1$, $V_1 + V_2$.

c) $W_i := \{p(t) \in \mathsf{Pol}_2(\mathbb{R}) \,|\, p(a_j)^{(j)} = 0 \text{ for } j \leqslant i\}$ with $i = 0, 1, 2$, are subspaces of $\mathsf{Pol}_2(\mathbb{R})$.

d) Give concrete descriptions of $W_0 \cap W_1 \cap W_2$ and $W_1 + W_2$.

Span, linear transformations, etc.

5) Let F^4 be the F-vector space as usual, and consider the system $(v_1, \ldots, v_5)$ given by $v_1 = (1, 1, 2, 4)$, $v_2 = (2, -1, -5, 2)$, $v_3 = (1, -1, -4, 0)$, $v_4 = (2, 1, 1, 6)$, $v_5 = (3, 2, 3, 10)$.

a) Find two maximal free subsystems of $(v_1, \ldots, v_5)$.

b) Which elements of the standard basis $\mathcal{E} = (e_1, \ldots, e_4)$ of F^4 lie in $\langle v_1, \ldots, v_5 \rangle_F$?

6) Let $v_1 = (2i, 1, 0)$, $v_2 = (2, -1, 1)$, $v_3 = (0, 1-i, 1-i)$ and $w_1 = (1, 0, -1)$, $w_2 = (1, 1, 1)$, $w_3 = (1, 0, 0)$ be vectors in $\mathbb{C}^3$.

a) Show that $\mathcal{A} := (v_1, v_2, v_3)$ and $\mathcal{B} = (w_1, w_2, w_3)$ are bases of $\mathbb{C}^3$.

b) Find the coordinates of $(1, 0, 1)$ and of (a, b, c) in each of the bases $\mathcal{A}$ and $\mathcal{B}$.

c) Find the base change matrix S from the basis $\mathcal{A}$ to basis $\mathcal{B}$, i.e., the matrix $S \in \mathbb{C}^{3\times 3}$ with $[v]_{\mathcal{B}} = S[v]_{\mathcal{A}}$ for all vectors $v \in \mathbb{C}^3$.

7) Let $V_1, V_2 \subset V$ be subspaces, and suppose that $\dim(V_1) = 3$, $\dim(V_2) = 4$, $\dim(V) = 5$.

a) What are the possible values for $\dim(V_1 + V_2)$ and $\dim(V_1 \cap V_2)$?

b) Show directly, i.e., without using the results from the class/books/etc., that every subspace $W \subseteq V$ is finitely generated, thus $\dim(W) \leqslant 5$.

8) Let $V \subset \mathcal{C}(\mathbb{R}, \mathbb{R})$ be the span of the system of functions $(f_1, \ldots, f_5)$, where $f_1 = \sin(x)$, $f_2 = \cos(x)$, $f_3 = \sin(2x)$, $f_4 = \sin^2(x)$, $f_5 = \cos^2(x)$. Define $T : V \to \mathcal{C}(\mathbb{R}, \mathbb{R})$ by $T(f) = f'$, where f' is the derivative of f.

a) Show that $\mathcal{A} := (f_1, \ldots, f_5)$ is a basis of V.

b) Show that T is a linear transformation, and compute $\ker(T)$ and $\operatorname{im}(T)$.

c) Find $A \in \mathbb{R}^{5\times 5}$ such that $(f_1', \ldots, f_5') = (f_1, \ldots, f_5)A$. Is the matrix A unique?

(∗) What is the relation between A and the matrix A_T of T in the basis $\mathcal{A}$?

d) Find all the solutions $f \in V$ of the differential equation $f'' - 2f' + f = 0$.

9) Let $\mathsf{Pol}_3(R)$ be the R-module of polynomials of degree $\leqslant 3$ over a ring R, and $a \in R$ fixed.

a) Prove/disprove that $\mathcal{A} := \big(1, t-a, (t-a)^2, (t-a)^3\big)$ is a basis of $\mathsf{Pol}_3(R)$.

b) Find a 4×4 matrix $P \in R^{4\times 4}$ such that $\mathcal{E} = \mathcal{A}P$, where $\mathcal{E} := (1, t, t^2, t^3)$ is the standard basis of $\mathsf{Pol}_3(R)$.

c) Let $T : \mathsf{Pol}_3(R) \to \mathsf{Pol}_3(R)$ be the linear transformation defined by $T(f) = tf' + f''$. Show that T is a linear transformation, and find its matrix in the basis $\mathcal{A}$.

10) For a ring R with $0_R \neq 1_R$, set $X := \{0_R, 1_R\}$, and $T : \mathsf{Pol}_n(R) \to \mathcal{F}(X, R)$ be defined by $T\big(p(t)\big) = f_{p(t)}$, with $f_{p(t)}(x) = p(x)$ for $x \in X$. Prove/disprove or answer the following:

a) T is a linear transformation. Describe $\ker(T)$ and $\operatorname{im}(T)$.

b) Let $\mathcal{A} = (1, t, t^2)$ be the standard basis of $\mathsf{Pol}_2(F)$ and $\mathcal{E} = (f_0, f_1)$ the standard basis of $\mathcal{F}(X, F)$. Find the matrix A of T with respect to the bases $\mathcal{A}$, $\mathcal{E}$.

c) The same question for $T : \mathsf{Pol}_n(R) \to \mathcal{F}(X, R)$ and $\mathcal{A} = (1, \ldots, t^n)$.