Due: Mo, Feb 7, 2022

Math 314 / Problem Set 2 (two pages)

1) Prove the assertions from the class:

a) The ideals of $\mathbb{Z}$ are of the form $m\mathbb{Z}$, $m \geq 0$.

b) For $m > 1$, and $\overline{a} \in \mathbb{Z}/m\mathbb{Z}$, prove the assertions form the class:

- $\overline{a}$ is a zero divisors iff a and m have a common divisor $r > 1$, i.e., a, m are not coprime.
- $\overline{a}$ is invertible iff a and m do not have a common divisor $r > 1$, i.e., a, m are coprime.

c) In particular, $\mathbb{Z}/m\mathbb{Z}$ is a field iff m is prime number.

d) List the zero divisors and the invertible elements in $\mathbb{Z}/30\mathbb{Z}$. What is the inverse of $\overline{7}$?

Notation. For p a prime number, $\mathbb{Z}/p\mathbb{Z}$ is denoted $\mathbb{F}_p$ and called the field with p elements.

2) Let X be a non-empty set, and $\mathcal{P}(X)$ be the set of subsets of X, and for $A, B \in \mathcal{P}(X)$, define the symmetric difference of the two subsets by $A \mathbin{\vartriangle} B := (A \backslash B) \cup (B \backslash A)$. Further, $\mathcal{F}(X, \mathbb{F}_2)$ is the ring of all the maps on X with values in he field with two elements $\mathbb{F}_2$. For $A \in \mathcal{P}(X)$, let $\chi_A \in \mathcal{F}(X, \mathbb{F}_2)$ be the characteristic function of A. Answer the following:

a) $\mathcal{P}(X), \Delta, \cap$ is a commutative ring. Solve the equation $x^2 + 1_{\mathcal{P}(X)} = 0_{\mathcal{P}(X)}$ in $\mathcal{P}(X)$.

b) The map $f : \mathcal{P}(X) \to \mathcal{F}(X, \mathbb{F}_2)$ by $f(A) := \chi_A$, is an isomorphism of rings.

c) Define $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\varphi(A) = \mathcal{C}_X(A) := X \backslash A$. Find an "addition" $*$ on $\mathcal{P}(X)$ such that $\mathcal{P}(X), *, \cup$ is a ring, and $\varphi : \mathcal{P}(X), \Delta, \cap \to \mathcal{P}(X), *, \cup$ is an isomorphism.

Modules & Vector spaces

3) Do the following:

a) For M be an R-module, review the proofs of the following assertion from the class:

- $r \cdot 0_M = 0_M = 0_R \cdot x \;\forall\, r \in R,\, x \in M$. If $r \in R^\times$, then $r \cdot x = 0_M$ iff $x = 0_M$.
- If V is an F-vector space, and $a \in F$, $v \in V$, then: $a \cdot v = 0_V$ iff $a = 0_F$ or $v = 0_V$.

b) Check the details of the proofs of the assertions from Example 2.4 from typed Notes.

4) Let R be a commutative ring with $0_R \neq 1_R$. Prove/answer the following:

a) A subset $M \subset R$ is an R-submodule of $R, +$ iff M is an ideal of R.

b) If R is a field if an only if the R-submodules of $R, +$ are $\{0_R\}$ and R itself.

c) An R-submodule M of $R, +$ is free iff $M = rR$ with $r \in R$ a non-zero divisor.

5) Prove/disprove the following:

a) $\mathbb{Q}, +$ is not a finitely generated $\mathbb{Z}$-module.

b) $\mathbb{R}, +$ is not a finitely generated $\mathbb{Q}$-vector space.

c) $R[t], +$ is not a finitely generated R-module.

6) Let M be an R-module, and $(x_i)_{i \in I}$ be a system of elements of M. Prove/answer the following (as mentioned in class):

a) If either $x_i = 0_M$ for some $i \in I$, or $x_i = x_j$ for some $i \neq j$, then $(x_i)_{i \in I}$ is not free.

b) Suppose that $I = \{1, \dots, n\}$, and that $a_1 x_1 + \dots + a_n x_n = 0_M$ for some $a_1, \dots, a_n \in R$ with a_n invertible. Then $\langle x_1, \dots, x_n \rangle_R = \langle x_1, \dots, x_{n-1} \rangle_R$.

$(*)$ Is the same true if a_n is not necessarily invertible?

7) Let $\mathcal{C}^r(I)$ be the set of all the real valued r-differentiable functions f on some open interval $I = (a, b)$ with $f^{(r)}$ continuous, and $\mathcal{C}_{\mathsf{Pol}}(I)$ be the set of all the real valued polynomial functions on I. Prove/disprove the following:

a) $\mathcal{C}_{\mathsf{Pol}}(I) \subset \mathcal{C}^r(I)$ are rings w.r.t. the usual addition and multiplication of maps.

b) $\mathcal{C}^r(I), +$ is not a finitely generated $\mathcal{C}_{\mathsf{Pol}}(I)$-module.

c) The differentiation map $f \mapsto f'$ on $\mathcal{C}^r(I)$ is not a morphism of $\mathcal{C}_{\mathsf{Pol}}(I)$-modules.

8) Let R be a commutative ring, X be an arbitrary non-empty set, and $\mathcal{F}(X, R)$ be the ring of all the R-valued maps on X (w.r.t to the usual $+$ and $\cdot$ of maps). Prove/disprove/answer:

a) $\mathcal{F}(X, R), +$ is finitely generated R-module if and only if X is finite.

b) If $X := \{x_1, \ldots, x_d\}$, give a basis of $\mathcal{F}(X, R), +$ as an R-module.

c) Let $\mathsf{Pol}_n(R) \subset R[t]$ be the R-module of polynomials of degree $\le n$. Then $\mathcal{F}(X, R)$ and $\mathsf{Pol}_n(R)$ are isomorphic R-modules iff $n = |X| + 1$.

Review of polynomial maps

Let R be a commutative ring with $1_R \neq 0_R$, and $R[t]$ be the polynomial ring over R. Suppose that $R \subset S$ are rings with $1_R = 1_S$ and $rs = sr$ for all $r \in R$, $s \in S$, but S is not necessarily commutative. For $p(t) \in R[t]$, defined the polynomial map $f_{p(t)} : S \to S$ defined by $f_{p(t)}(b) := p(b)$ for all $b \in S$.

9) Recalling the ring of abstract maps $\mathcal{F}(S, S)$ with the usual $+, \cdot$, prove the following:

a) The map $\varphi : R[t] \to \mathcal{F}(S, S)$ defined by $p(t) \mapsto f_{p(t)}$ is a ring homomorphism.

b) In the case $R = \mathbb{Z}/6\mathbb{Z} = S$, find $\mathrm{Ker}(\varphi)$. Does $f_{p(t)}$ always determine $p(t)$?

c) Prove or disprove that the following assertions are equivalent:

i) φ is injective, i.e., for all $p(t), q(t) \in R[t]$ one has: $f_{p(t)} = f_{q(t)}$ iff $p(t) = q(t)$.

ii) S is infinite.

10) For every fixed $a \in R$, consider the map $\varphi_a : R[t] \to R$ by $\varphi_a\big(p(t)\big) = p(a)$.

a) Show that $\mathrm{Ker}(\varphi_a) = (t - a)R[t]$, i.e., $\varphi_a\big(p(t)\big) = p(a) = 0$ if and only if $p(t)$ is divisible by $t - a$ in $R[t]$.

b) For $a, b \in R$, prove/disprove: $p(a) = 0_R = p(b)$ iff $(x - a)(x - b)$ divides $p(t)$ in $R[t]$.