Due: Mo, May 2, 2022

Math 314 / Problem Set 12 (two pages)

- **Study/read:** *Bilinear/quadratic forms*
  - Ch. 7 of *LADW* by Treil;
  - Ch. 10 of *Linear Algebra* by Hoffman & Kunze.

Bilinear forms & Quadratic forms

- For V a F-vector space, recall:
  - $\mathcal{L}^2_{\text{sym}}(V,F), \mathcal{L}^2_{\text{alt}}(V,F) \subset \mathcal{L}^2(V,F)$ are subspaces of the F-vector space $\mathcal{L}^2(V,F)$.
  - $\varphi \in \text{End}_F(V)$ is an automorphism of $f \in \mathcal{L}^2(V,F)$, if $f(\varphi \boldsymbol{x}, \varphi \boldsymbol{y}) = f(\boldsymbol{x}, \boldsymbol{y})\ \forall\ \boldsymbol{x}, \boldsymbol{y} \in V$.
  - For $\mathcal{A} = (\alpha_1, \dots, \alpha_m)$ a basis of V, one has a canonical isomorphism of F-vector spaces:

$$\Psi_{\mathcal{A}} : \mathcal{L}^2(V,F) \to F^{\,m\times m} \ \text{ defined by } \ \Psi(f) := A_f := \Big(f(\alpha_i, \alpha_j)\Big)_{i,j}$$

- For $\boldsymbol{x} \in V$, let $[\boldsymbol{x}] := [\boldsymbol{x}]_{\mathcal{A}} \in {}^mF$ be the coordinate vector of $\boldsymbol{x}$. Make sure that you know/checked the details of the proofs assertions from the class:
  - For $\boldsymbol{x}, \boldsymbol{y} \in V$ one has $f(\boldsymbol{x}, \boldsymbol{y}) = [\boldsymbol{x}]^\tau A_f\, [\boldsymbol{y}]$.
  - $\Psi_{\mathcal{A}} : \mathcal{L}^2(V,F) \to F^{\,m\times m}$ is an isomorphisms of F-vector spaces.
  - f is symmetric / alternating iff A_f is so.
  - If $f \in \mathcal{L}^2_{\text{sym}}(V,F)$ or $f \in \mathcal{L}^2_{\text{alt}}(V,F)$, then $\text{Ker}_l(f) =: \text{Ker}(f) := \text{Ker}_r(f)$.
    In general, one has $\dim\Big(\text{Ker}_l(f)\Big) = \dim\Big(\text{Ker}_r(f)\Big)$.
  - If $\mathcal{B} = \mathcal{A}\, S$ with $S = S_{\mathcal{A}\mathcal{B}}$ the matrix of change of basis from $\mathcal{A}$ to $\mathcal{B}$, then

$$\Psi_{\mathcal{B}} = S^\tau \Psi_{\mathcal{A}}\, S, \ \text{ i.e., if } \ B_f := \Psi_{\mathcal{B}}(f), \text{ then } \ B_f = S^\tau A_f\, S.$$

- Let $\frac{1}{2} \in F$, i.e., $2 \neq 0_F$, and $\mathcal{A}$ be a basis of V. For $f \in \mathcal{L}^2_{\text{sym}}(V,F)$ and $A_f = \Psi_{\mathcal{A}}$, let $q_f(\underline{\boldsymbol{X}}) := \underline{\boldsymbol{X}} A_f \underline{\boldsymbol{X}}^\tau$ be the corresponding quadratic form, where $\underline{\boldsymbol{X}} = (X_1, \dots, X_m)$. (WHY).

Terminology: V endowed with $q_f(\boldsymbol{x}) := f(\boldsymbol{x}, \boldsymbol{x})$ for $\boldsymbol{x} \in V$ is called a quadratic space. Hence if V, q_f is a quadratic space, then $q_f(\underline{\boldsymbol{X}}) := \underline{\boldsymbol{X}}\, A_f \underline{\boldsymbol{X}}^\tau$ is a quadratic form in the usual sense.

1) Prove the polarization identity: $f(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4}\Big[q_f(\boldsymbol{x}+\boldsymbol{y}) - q_f(\boldsymbol{x}-\boldsymbol{y})\Big], \quad \boldsymbol{x}, \boldsymbol{y} \in V$.
In particular, $f = 0$ iff $f(v,v) = q_f(v) = 0$ for all $v \in V$.

2) Let $\boldsymbol{x} = (x_1, \dots, x_m)$, $\boldsymbol{y} = (y_1, \dots, y_m)$ denote elements of F^m. Answer the following:

a) Find the matrix A_f of the bilinear form $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}\, A_f\, \boldsymbol{y}^\tau$ on F^3 define by:
$f(\boldsymbol{x}, \boldsymbol{y}) = x_1y_1 + 2x_1y_2 + 14x_1y_3 - 5x_2y_1 + 2x_2y_2 - 3x_2y_3 + 8x_3y_1 + 19x_3y_2 - 2x_3y_3$.

b) For $\boldsymbol{x}, \boldsymbol{y} \in F^2$, let $A_{\boldsymbol{x},\boldsymbol{y}}$ be the matrix with rows $\boldsymbol{x}, \boldsymbol{y}$. Prove that $f(\boldsymbol{x}, \boldsymbol{y}) = \det(A_{\boldsymbol{x},\boldsymbol{y}})$ is bilinear on $F^2 \times F^2$, and compute the matrix A_f of f.

c) Find the symmetric matrix A_q of the quadratic form $q(\boldsymbol{x})$ on F^3 in the case
$q(\boldsymbol{x}) = x_1^2 + 2x_1x_2 - 3x_1x_3 - 9x_2^2 + 6x_2x_3 + 13x_3^2$.

3) For $A \in F^{m\times m}$ as below, diagonalize the resulting quadratic form $q_A(\underline{\boldsymbol{X}}) = \underline{\boldsymbol{X}} A \underline{\boldsymbol{X}}^\tau$ both abstractly, and othonormally in the case $F = \mathbb{R}$:

$$\text{a)} \quad A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix} \qquad \text{b)} \quad A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

• Recall that $A \in \mathbb{R}^{m\times m}$ is called (semi) positively definite, if the quadratic form $q_A : \mathbb{R}^m \to \mathbb{R}$ takes values > 0 (respectively ≥ 0). One defines correspondingly (semi) negative definite. One says that A is indefinite, if q_A is neither (semi) positive or (semi) negative definite.

4) Let $A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{pmatrix}$ in $\mathbb{R}^{3\times 3}$ be given.

Using *Sylvester's law of inertia,* check if $\pm A$, $\pm B$, $A \pm B$, $A^2 \pm B^2$ are (semi)positive definite.

5) Let $q = \underline{\boldsymbol{X}} A \underline{\boldsymbol{X}}^\tau$ and $\tilde{q} = \underline{\boldsymbol{X}} \tilde{A} \underline{\boldsymbol{X}}^\tau$ be quadratic forms with symmetric matrices A, respectively $\tilde{A}$. Suppose that q is positive definite. True or false (justify if not obvious):

a) If $\tilde{A} := A^5$, then $\tilde{q}$ is positive definite.

b) If $\tilde{A} := -A^8$, then $\tilde{q}$ is negative definite.

d) If $\tilde{q}$ is negative semidefinite, then the matrix $A - \tilde{A}$ is positive definite.

e) If $\tilde{q}$ is indefinite, then the matrix $A + \tilde{A}$ is positive definite.

6) For $f \in \mathcal{L}^2(V, F)$, let $\mathrm{Aut}(f) \subset \mathrm{End}_F(V)$ be the set of automorphisms of f, and for a fixed basis $\mathcal{A}$ of V, set $A := \Psi_{\mathcal{A}}(f) = \big(f(\alpha_i, \alpha_j)\big)_{i,j}$. Prove/disprove:

a) $\mathrm{Aut}(f) \subset \mathrm{End}_F(V)$ is a submonoid of $\mathrm{End}_F(V), \circ$.

b) $\varphi \in \mathrm{Aut}(f)$ iff $[\varphi]_{\mathcal{A}}^\tau A\,[\varphi]_{\mathcal{A}} = A$, i.e., $\Psi_{\mathcal{A}}\big(\mathrm{Aut}(f)\big) = \{\, S \in F^{m\times m} \mid S^\tau A S = A \,\}$.

c) The following assertions are equivalent:

i) $\mathrm{Aut}(f) \subset \mathrm{End}_F(V)$ is a group w.r.t. composition of maps.

ii) $\{\, S \in F^{m\times m} \mid S^\tau A S = A \,\} \subset \mathrm{GL}_m(F)$ is a subgroup.

iii) $\mathrm{Ker}_l(f) = \mathrm{Ker}_r(f) = \{0_V\}$.

[**Hint:** To 1): One has $[\varphi(\boldsymbol{x})]_{\mathcal{A}} = S\,[\boldsymbol{x}]_{\mathcal{A}}$, hence $A_{f\circ\varphi^2} = S^\tau A_f S$ (WHY), and $\varphi \in \mathrm{Auf}(f)$ iff $A_f = S^\tau A_f S$ (WHY), etc.…
To 2): i)⇔ii) is clear (WHY); ii)⇒iii): By contradiction: if $0_V \neq x \in \mathrm{Ker}_l(f)$, choose $\mathcal{A}$ s.t. $\alpha_1 = x$, and $S \in F^{m\times m}$ to be $S := \boldsymbol{I}_m - \boldsymbol{e}_{11}$. Then $S^\tau A_f S = A$ (WHY), but $S \notin \mathrm{GL}_m(F)$, (WHY); iii)⇒i): $x \in \mathrm{Ker}(\varphi) \Rightarrow x \in \mathrm{Ker}_l(f), \mathrm{Ker}_r(f)$ (WHY)…]

Supplement: Prove the following:

Proposition. *For every bilinear alternating $f \in \mathcal{L}^2_{\mathrm{alt}}(V, F)$ there exists a basis $\mathcal{A}$ of V and $d, e \geq 0$ such that $\dim(V) = 2d + e$ with $e = \dim\big(\mathrm{Ker}(f)\big)$, and $A_f := A_{f,\mathcal{A}}$ is of the form:*

$$A_f = \begin{pmatrix} \mathbf{0}_{e\times e} & \mathbf{0}_{e\times 2d} \\ \mathbf{0}_{2d\times e} & B \end{pmatrix}, \quad B = \big(B_{kl}\big)_{k,l} \text{ with } B_{kk} = \begin{pmatrix} 0 & a_k \\ -a_k & 0 \end{pmatrix}, \quad B_{kl} = \mathbf{0}_{2\times 2} \text{ for } k \neq l.$$

[**Hint:** If $f \neq 0$, $\exists\, v_1, v_2$ lin. indep. s.t. $f(v_1, v_2) \neq 0$ (WHY). Set $W_1 := \langle v_1, v_2\rangle_F$, $W := W_1^\perp$. Then $V = W_1 + W$, etc.…]