Due: Mo, Apr 25, 2022

Math 314 / Problem Set 11 (two pages)

- **Study/read** *Inner product spaces*:

 - Ch. 6, especially sections 3, 5, of *LADW* by Treil;
 - Ch. 8, 9 of *Linear Algebra* by Hoffman & Kunze.

(!) Make sure that you understand perfectly the definitions, examples, theorems, etc.

Check whether you know the basics:

- **True or false**:

 a) Every unitary operator is normal.
 b) A matrix is unitary if and only if it is invertible.
 c) If two matrices are unitarily equivalent, then they are also similar.
 d) The sum of self-adjoint operators is self-adjoint.
 e) The adjoint of a unitary operator is unitary.
 f) The adjoint of a normal operator is normal.
 g) If all eigenvalues of a linear operator are 1, then the operator is unitary or orthogonal.
 h) If all eigenvalues of a normal operator are 1, then the operator is identity.
 i) A linear operator may preserve norm but not the inner product.

- **True or false**:

 a) The polar decomposition of a matrix $A = U|A|$ is unique.
 b) $\lambda \in F$ is eigenvalue of A iff $|\lambda|$ is eigenvalues of $|A|$.
 c) Singular values of a matrix are also eigenvalues of the matrix.
 d) Singular values of a matrix A are eigenvalues of AA^*.
 e) Is σ is a singular value of a matrix A and $c \in F$, then $c\sigma$ is a singular value of cA.
 f) The singular values of any linear operator are non-negative.
 g) Singular values of a self-adjoint matrix coincide with its eigenvalues.
 h) $|\det(A)| = \det(|A|)$.

• Solve the problems below:

1) Find unitary matrices U such that UAU^{-1} is diagonal in the following cases:

$$\text{a) } A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{b) } A = \begin{pmatrix} 0 & 1+i \\ 1-i & 1 \end{pmatrix} \quad \text{c) } A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

2) Consider the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}$.

 a) Find all square roots of A, i.e., self-adjoint semi-positive matrices X_0 with $X_0^2 = A$.
 b) Find all the matrices $X \in \mathbb{R}^{3\times 3}$ such that $X^2 = A$.

3) Let $A \in F^{n\times n}$ with $F = \mathbb{R}$ or $F = \mathbb{C}$ be given. Prove/disprove the following assertions:

a) $I_n + AA^*$ is invertible.

b) If A is self-adjoint, then $I_n + A$ is invertible.

c) If A is unitary, then $A + cI_n$ is invertible for $0 < c < 1$.

d) If $A \in \mathbb{R}^{n\times n}$, then $B := A + iI_n \in \mathbb{C}^{n\times n}$ is invertible.

4) Answer/prove/disprove the following:

a) Let $T : V \to V$ be both positive and unitary. Then $T = I$.

b) Let $T_1, T_2 : V \to V$ be normal operators which commute, i.e., $T_1T_2 = T_2T_1$. Then $T_1 \pm T_2$ and T_1T_2 are normal operators.

5) Let $T : V \to V$ be a normal operator, where $\dim_{\mathbb{C}}(V) < \infty$. Prove/disprove the following:

a) If all the eigenvalues λ satisfy $|\lambda| = 1$, then T is unitary.

b) If all the eigenvalues of T are real, then T is self-adjoint.

6) Find polar decompositions $A = U|A|$ for the following matrices:

$$\text{a) } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{b) } A = \begin{pmatrix} 0 & 1+i \\ -i & 1 \end{pmatrix} \quad \text{c) } A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

7) For $F = \mathbb{R}, \mathbb{C}$, let $A \in F^{m\times m}$ be invertible, and $A = S\,\Sigma\,T$ be a singular value decomposition of A. Find a singular value decompositions of A^* and of A^{-1}.

8) Let V be a finite dimensional F-vector space with inner product over $F = \mathbb{C}$, and $T \in \mathrm{End}_F(V)$ be a self-adjoint operator. Prove/disprove the following:

a) $\|v + iT(v)\| = \|v - iT(v)\|$ for all $v \in V$.

b) $v + iT(v) = w + iT(w)$ if and only if $v = w$.

c) $I + iT$ and $I - iT$ are isomorphisms, hence $(I + iT)^{-1}$ and $(I - iT)^{-1}$ exist.

d) $U := (I - iT)(I + iT)^{-1}$ is a unitary operator.

Terminology: The above operator U is called the Cayley transform of T. Notice that the map $T \mapsto U := (I - iT)(I + iT)^{-1}$ is similar to the map $t \mapsto z := \frac{1-ti}{1+ti}$ from $\mathbb{R}$ into the unit circle $\big(|z| = 1\big)$ in $\mathbb{C}$. **Note** that one has indeed $\left|\frac{1-ti}{1+ti}\right| = 1$ for all $t \in \mathbb{R}$ (WHY).
Google the term "Cayley transform" and learn more about it.

Supplement*: Let V be an $\mathbb{R}$-vector space with inner product. Prove/disprove:

a) The unitary operators $T : V \to V$ are precisely the orthogonal ones.

b) Let $T : V \to V$ be unitary and $T^2 = -\mathrm{id}_V$. Then $T^* = -T$ and $\dim(V)$ is even, say $\dim(V) = 2n$. Further, there exists a subspace $E \subset V$ with $\dim(E) = n$ satisfying:
- $T(E) = E^\perp$ and $T(E^\perp) = E$.
- If $\mathcal{E} = (v_1, \ldots, v_n)$ is an orthonormal basis of E, then $\mathcal{E}^\perp := (Tv_1, \ldots, Tv_n)$ is an orthonormal basis for $E^\perp$ and so is $\mathcal{A} := (\mathcal{E}, \mathcal{E}^\perp)$ for V.
- $[T]_{\mathcal{A}} = \begin{pmatrix} \mathbf{0}_m & U^* \\ U & \mathbf{0}_m \end{pmatrix}$ with $U \in \mathbb{R}^{m\times m}$ unitary.