Due: Fr, Apr 15, 2022

Math 314 / Problem Set 10 (two pages)

- **Study/read:** *Inner product spaces & Operators on inner product spaces*

- Ch. 5 & 6 from *LADW* by Treil, Ch. 8 & 9 from *Linear Algebra* by Hoffman & Kunze.

(!) Make sure that you understand perfectly the definitions, examples, theorems, etc.

- Make sure that you check the details of the proofs of the following.

I) Let V be an F-v.s., $\mathcal{A} = (\alpha_i)_{1\leqslant i\leqslant m}$ be a basis, $(\,|\,)$ be the standard inner product on on ${}^mF = F^{m\times 1}$. Hence if $\boldsymbol{x}, \boldsymbol{y} \in F^{m\times 1} = {}^mF$, then $\boldsymbol{y}^* \in F^{1\times m} = F^m$ and $(\boldsymbol{x}\,|\,\boldsymbol{y}) = \boldsymbol{y}^*\boldsymbol{x}$ (WHY).

- If $v, w \in V$, $\boldsymbol{x} = [v]_{\mathcal{A}}$, $\boldsymbol{y} = [w]_{\mathcal{A}}$, then $(v\,|\,w)_{\mathcal{A}} = (\boldsymbol{x}\,|\,\boldsymbol{y})$.
- If $(\,|\,)$ is given on V, then $(\,|\,) = (\,|\,)_{\mathcal{A}}$ on V iff $\mathcal{A}$ is orthonormal.

II) Let $c \in F$, and suppose that the adjoints of $f, f_1, f_2 : W \to V$, $g : V \to U$ exist. TFH:

- $(cf)^*$ and $(f^*)^*$ exist and $(cf)^* = \bar{c}f^*$, $(f^*)^* = f$.
- $(f_1 + f_2)^*$ and $(g \circ f)^*$ exist and $(f_1 + f_2)^* = f_1^* + f_2^*$, $(g \circ f)^* = f^* \circ g^*$.
- $\mathrm{Ker}(f) = \mathrm{Im}(f^*)^\perp$, $\mathrm{Ker}(f^*) = \mathrm{Im}(f)^\perp$, $\mathrm{Im}(f) = \mathrm{Ker}(f^*)^\perp$, $\mathrm{Im}(f^*) = \mathrm{Ker}(f)^\perp$
- $\mathrm{Ker}(f^* \circ f) = \mathrm{Ker}(f)$, $\mathrm{Ker}(f \circ f^*) = \mathrm{Ker}(f^*)$.

III) For $A \in F^{m\times n}$ and $T_A \colon {}^nF \to {}^mF$, $\boldsymbol{x} \mapsto A\boldsymbol{x}$, and $A^* \in F^{n\times m}$ and T_{A^*} one has:

- $(A \pm B)^* = A^* \pm B^*$; $aA^* = \bar{a}A$ for all $a \in F$; $(AB)^* = B^*A^*$; $(A^*)^* = A$.
- $(A\boldsymbol{x}\,|\,\boldsymbol{y}) = (\boldsymbol{x}\,|\,A^*\boldsymbol{y})\ \forall\, \boldsymbol{x} \in {}^nF$, $\boldsymbol{y} \in {}^mF$, where $(\,|\,)$ is the standard inner product.
- $\mathrm{Ker}(T_A) = \mathrm{Im}(T_{A^*})^\perp$, $\mathrm{Im}(T_A) = \ker(T_{A^*})^\perp$, $\mathrm{Ker}(T_{A^*}) = \mathrm{Im}(T_A)^\perp$, $\mathrm{Im}(T_{A^*}) = \mathrm{Ker}(T_A)^\perp$.
- $\mathrm{Ker}(T_A) = \ker(T_{A^*A})$ and $\ker(T_{A^*}) = \ker(T_{AA^*})$.

IV) For $A \in \mathbb{R}^{n\times n}$ and $T_A : {}^n\mathbb{R} \to {}^n\mathbb{R}$ by $T_A(\boldsymbol{x}) = A\boldsymbol{x}$, one has:

- $\ker(T_A)^\perp = \mathrm{Im}(T_{A^\tau}$ and $\mathrm{Im}(T_A) = \mathrm{Ker}(T_{A^\tau})^\perp$ inside ${}^n\mathbb{R}$.
- What is the corresponding assertion for $A \in \mathbb{C}^{n\times n}$?

1) Which of the following pairs of matrices are unitarily similar (or equivalent):

a) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ & $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ b) $\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ & $\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$ c) $\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ & $\begin{pmatrix}1 & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & i\end{pmatrix}$

[**Hint:** Unitarily equivalent matrices have the same characteristic polynomial (WHY), thus the same eigenvalues including multiplicities, the same trace, determinant, etc.]

2) For each of the following matrices over $F = \mathbb{R}$ decide whether they are unitarily diagonalizable, and if so, find the unitary matrix P over $\mathbb{R}$ such that PAP^{-1} is diagonal.

a) $A = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ b) $A = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ c) $A = \begin{pmatrix}\cos x & 0 & \sin x\\ 0 & -2 & 0\\ \sin x & 0 & -\cos x\end{pmatrix}$ d) $A = \begin{pmatrix}0 & 1 & 2 & 3\\ 1 & 2 & 3 & 4\\ 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 6\end{pmatrix}$

3) In each of the matrices below give all the values of the entries such that the matrix are:
a) unitary; b) self adjoint; c) normal; d) orthogonal

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & a_{13} \\ a_{21} & a_{22} & \frac{1}{3} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & \frac{1+i}{2} & a_{13} & \frac{1}{2} \\ a_{12} & a_{22} & \frac{1-i}{2} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

• In the next problems, if not otherwise explicitly stated, V is a finite dimensional F-vector space with inner product, where $F = \mathbb{R}$ or $F = \mathbb{C}$.

4) Let $A, B.S \in F^{n\times n}$. True of false (justify if not obvious):
a) If $A, B \neq 0_{n\times n}$ are self-adjoint, so are $2A - 3B$, $A - (1+i)B$, $xA + yB$ for all $x, y \in F$.
b) If $A, B \neq 0_{n\times n}$ are self-adjoint, so are A^5, AB, $(A \pm B)^2$.
c) If A, B are unitarily equivalent, i.e., $B = SAS^{-1}$ for some unitary matrix S, then:
(i) A is unitary iff B is so. (ii) A is self-adjoint iff B is so. (iii) A is normal iff B is so. (iv) A is (semi-)positive iff B is so. (v) A is diagonalizable iff B is so.
d) The same questions in the case A, B are orthogonally equivalent, i.e., $B = SAS^{-1}$ for an orthogonal matrix S. (The answer might depend on whether $F = \mathbb{R}$ or $F = \mathbb{C}$.)
e) The same questions in the case A, B are abstractly equivalent, i.e., $B = SAS^{-1}$ for some invertible matrix S. (The answer might depend on whether $F = \mathbb{R}$ or $F = \mathbb{C}$.)

5) For each of the matrices $A \in F^{n\times n}$ below find U, Δ *decompositions* $A = U\Delta$ (i.e., with $U \in F^{n\times n}$ unitary and $\Delta \in F^{n\times n}$ upper triangular with non-negative diagonal entries) and *polar decompositions* $A = UN$ (i.e., with U unitary and N (semi)positive definite).

$$\text{a)}\ A = \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix} \qquad \text{b)}\ A = \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ i & 0 & 1 \end{pmatrix} \qquad \text{c)}\ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

More about projections: Let V be an F vector space with inner product.

6) Suppose that $\dim(V) = n$, $E \subset V$, and $E^\perp \subset V$ is the orthogonal subspace to E.
a) Let $P \in F^{n\times n}$ be the matrix of the orthogonal projection $pr_E : V \to E$ in an orthonormal basis $\mathcal{A}$ of V. Then P is self-adjoint and $P^2 = P$.
b) Conversely, if $P \in F^{n\times n}$ is self-adjoint and $P^2 = P$, then the transformation $T : V \to V$ with $[T]_\mathcal{A} = P$ is a orthogonal projection onto some subspace $E \subset V$.
$*$) Give a concrete description of E in terms of P and $\mathcal{A}$.

7) Let $E \subset V$ a subspace, and suppose that the orthogonal projections $P : V \to V$ and $Q : V \to V$ onto E, respectively $E^\perp$ exist, that is, P, Q are defined. Prove/disprove/answer:
a) For all $v \in V$ one has $v = P(v) + Q(v)$, i.e. $P + Q = I$ is the identity map of V.
b) For all $v \in V$ one has $P\big(Q(v)\big) = 0_V = Q\big(P(v)\big)$, i.e., $P \circ Q = \mathbf{0}_{\mathrm{End}(V)} = Q \circ P$.
c) $T_{a,b} := aP + bQ$ is invertible if and only if $a, b \neq 0$. What is the inverse of $T_{a,b}$?