Due: Fr, Jan 28, 2022

Math 314 / Problem Set 1 (two pages)

Composition laws

1) Let $I = [0,1]$ or $I = (0,1)$, and $A \in \mathbb{R}^{2\times 2}$ be 2×2 matrices. In each of the following cases determine the largest monoid, respectively group in the specified set:

- The interval I endowed with the usual addition, respectively multiplication.
- $\mathcal{C}^{\leqslant} := \{f : I \to I \,|\, f \text{ continuous increasing } \}$ w.r.t. $+$, resp. multiplication of functions.
- The set of all the antisymmetric matrices $A \in \mathbb{R}^{2\times 2}$ w.r.t. multiplication.
- The set of all the matrices $A \in \mathbb{R}^{2\times 2}$ having $\det(A) > 0$ w.r.t. multiplication.
- The set of matrices $A \in \mathbb{R}^{2\times 2}$ having even integer coefficients w.r.t. addition.
- The set of matrices $A \in \mathbb{R}^{2\times 2}$ with non-negative coefficients w.r.t. multiplication.
- The set of all the reflections about lines trough the origin w.r.t. composition of maps.
- $\mathcal{F}(X) := \{f : X \to X \,|\, f \text{ arbitrary map } \}$ w.r.t. map composition.
- The unit circle $\mathbb{S} := \{z \in \mathbb{C} \,\big|\, |z| = 1\}$ w.r.t. multiplication.
- $\mathcal{S} := \{f : \mathbb{R} \to \mathbb{R} \,|\, f(x) = ax + b,\ a, b \in \mathbb{R},\ a \neq 0\}$ w.r.t. composition $\circ$ of maps.
- $\{2^m \,|\, m \in \mathbb{Z}\}$ w.r.t. multiplication.

2) Let X be a non-empty set. The symmetric difference on $\mathcal{P}(X) := \{A \mid A \subseteq X\}$ is defined by $A \,\Delta\, B := (A \backslash B) \cup (B \backslash A)$. Prove that $\mathcal{P}(X)$ endowed with Δ is an abelian group.

3) Let S_5 be the permutations group of $\{1, 2, 3, 4, 5\}$, and consider $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ as elements of S_5. Solve the following equations in S_5 for the unknown x:

a) $x \circ \sigma = \tau$.

b) $x^2 \circ \sigma = \tau$, respectively $x \circ \sigma \circ x = \tau$.

4) Describe as permutation groups of the vertices of the following transformation groups:

a) The transformation group $\mathcal{T}_{ABCD}$ of the square $ABCD$.

b) The transformation group of the regular pentagon $ABCDE$.

c) The transformation group of the cube $ABCDA'B'C'D'$ which fix the vertices A, A'.

5) Let $M, *$ be a monoid with neutral element $e \in M$. Prove/disprove the following:

a) $G := \{g \in M \,|\, x \text{ has an inverse in } M\}$ endowed with $*$ is a group.

b) If for every $x \in M$ there exits $x' \in M$ such that $x' * x = e$, then $M, *$ is a group.

6) Let $R, +, \cdot$ be a commutative ring with $0_R \neq 1_R$. Recall that the set of invertible elements of R is $R^\times := \{x \in R \,|\, x \text{ invertible w.r.t to multiplication}\}$. Prove/disprove:

a) All $r \in R^\times$ are not zero divisors, i.e., for all $x \in R$, $x \neq 0_R$, one has that $rx \neq 0_R$.

b) $R^\times$ is a group with respect to the multiplication.

c) For every $r \in R$, the set $rR := \{rx \mid x \in R\}$. Then one has:

- $r_1 R = r_2 R$ if and only if there exists $x \in R^\times$ such that $r_2 = x r_1$.
- $rR = R$ if and only if r is invertible.

(Cartesian) products of algebraic structures

Let $*'$ and $*''$ be composition laws on X', respectively X''. Define the coordinate wise composition law $* := *' \times *''$ on $X := X' \times X''$ by $(x', x'') * (y', y'') := (x' *' y',\ x'' *'' y'')$.

7) Prove/disprove:

a) $*$ is associative, reps. commutative if and only if $*'$ and $*''$ are so.

b) $*$ has a neutral element e iff $*'$ and $*''$ have neutral elements e', e''.

c) $x := (x', x'')$ is invertible iff x' and x'' are invertible.

8) Let $G := G' \times G''$, $R := R' \times R''$ and $* = *' \times *''$, $\circ = \circ' \times \circ''$. Prove the following:

a) *G', $*'$ and G'', $*''$ are (abelian) monoids, resp. groups, iff $G, *$ is so.*

b) *R', $*'$, $\circ'$ and R'', $*''$, $\circ''$ are (commutative) rings iff $R, *, \circ$ is so.*

($\bullet$) **Question:** Is the same true for fields R', R'' ?

The monoid/group/ring/R-module of functions

Let X, T be non-empty sets, and $\mathrm{Maps}(X, T) := \{f \mid f : X \to T \text{ map}\}$, and suppose that $\cdot$ is a composition law on T. Then there is a canonically defined composition law $\bullet$ on $\mathrm{Maps}(X, T)$, defined by $(f \bullet g)(x) := f(x) \cdot g(x)$ for all $x \in X$.

9) In the above notation, prove/disprove the following:

a) $\bullet$ is associative, respectively commutative iff $\cdot$ is so.
Further, $\cdot$ has a neutral element e iff $\bullet$ does so. What is $e_\bullet$ as a function?

b) $f \in \mathrm{Maps}(X, T)$ is invertible w.r.t. $\bullet$ iff $f(x) \in T$ is invertible w.r.t. $\cdot$ for all $x \in X$.

c) $G, \cdot$ is an (abelian) monoid, respectively group iff $\mathrm{Maps}(X, G), \bullet$ is so.

d) $R, +, \cdot$ is a (commutative) ring with 1_R iff the corresponding $\mathrm{Maps}(X, R)$, $\boldsymbol{+}$, $\bullet$ is so.

e) $M, +$ is an R-module iff $\mathrm{Maps}(X, M)$, $\boldsymbol{+}$ is so.

($\bullet$) **Question:** Is the same true for (skew) fields R?

Language: $\mathrm{Maps}(X, T)$ is the monoid/group/ring/R-module of T-valued functions on X.