Due: Fr, Jan 21, 2022

Math 314 (Advanced Liner Algebra)
Prerequisites HW (three pages)

Recall that to disprove an assertion means to *give an example* where the assertion is not true.
Example. Prove/disprove that:

a) For all $x, y \in \mathbb{R}$ there exists $z \in \mathbb{R}$ such that $z^2 = x^2 + y^2$.
b) The same question for $x, y, z \in \mathbb{Q}$.

Solution. The assertion a) is true, because for $x, y \in \mathbb{R}$ one has: $x^2, y^2 \geqslant 0$, thus $x^2 + y^2 \geqslant 0$. Further, $\alpha := x^2 + y^2 \geqslant 0$ in $\mathbb{R}$, hence $\exists\, z \in \mathbb{R}$ s.t. $z^2 = \alpha$ (WHY).
The assertion b) is wrong, because $x = 1 = y$ implies that $x^2 + y^2 = 2$, and there is no $z \in \mathbb{Q}$ such that $z^2 = 2$, because $\sqrt{2} \notin \mathbb{Q}$ (WHY).

Basics of logical deduction

Recall that given assertions p, q, one define their (logical) disjunction $p \vee q$, (logical) conjunction $p \,\&\, q$, and (logical) negation $\neg\, p$.

1) Using the table of truth, prove the following assertions:

(i) $\neg(p \vee q)$ is the same as $\neg p \,\&\, \neg q$. (ii) $\neg(p \,\&\, q)$ is the same as $\neg p \vee \neg q$.

2) Show that using parentheses is essential for building desired unambiguous assertions. Namely, the assertion $p \vee q \,\&\, r$ is ambiguous. Indeed, the possible interpretations are

(i) $(p \vee q) \,\&\, r$ (ii) $p \vee (q \,\&\, r)$

and prove that the assertions (i) and (ii) are not equivalent in general.

3) Using the table of truth, prove the following properties of $\vee$ and &:

a) $\vee$ and & are associative, i.e., one has:
$(p \vee q) \vee r$ is the same as $p \vee (q \vee r)$, and $(p \,\&\, q) \,\&\, r$ is the same as $p \,\&\, (q \,\&\, r)$.
b) & is distributive w.r.t. $\vee$, i.e., one has: $(p \vee q) \,\&\, r$ is the same as $(p \,\&\, r) \vee (q \,\&\, r)$.
c) Is $\vee$ distributive with respect to &?

Using the quantifiers. Recall the (universal) quantifier $\forall$, the (existential) quantifier $\exists$, their usage, e.g. their negations: If $p(x)$ is an assertion depending on x, then one has:

(i) $\neg\big(\forall\, x\ p(x)\big)$ is the same as $\exists\, x\,\big(\neg\, p(x)\big)$; (ii) $\neg\big(\exists\, x\ p(x)\big)$ is the same as $\forall\, x\,\big(\neg\, p(x)\big)$.

4) In the set of natural numbers $\mathbb{N} = \{0, 1, \dots\}$, consider the assertion in plain English:

$(*)$ *Every natural number less than 2022 is a sum of three squares of natural numbers.*

a) Write the above assertion using quantifiers.
b) What is the negation of the above assertion in plain English.
c) Write the negation of the above assertion using quantifiers.
d) Is the above assertion true?

5) Recall that a real number x is a square of a real number iff $x \geqslant 0$. Consider the implication:

$$x, y \in \mathbb{R} \Rightarrow \exists\, z \in \mathbb{R} \quad \text{s.t.} \quad x^2 + y^2 = z^2$$

a) Formulate the above implication as an assertion in plain English.
b) Write the negation of the above implication, both with qualifiers, and in plain English.
c) Prove that the implication above is true, both directly, and arguing by contradiction.

Sets, correspondences, maps, relations (Google it!/check **Prereq LA**).

Recall that we always work in (Zermelo-Fraenkel System of Axioms & Axiom of Choice) ZFC. Spend some time trying to get used to the axiomatic way of thinking & working with sets!!!

6) Let A, B, C, D be given sets, and x be elements, e.g., real numbers. Answer the following:

a) Using $\cup$, $\cap$, $\setminus$ and A, B, C, D write down the sets of all x which satisfy:
i) $\big(x \in A \text{ or } x \in B\big)$ & $x \in C$ & $x \notin D$; ii) $x \in A$ or $\big(x \in B$ & $x \in C\big)$ & $x \notin D$.

b) Write as a union of disjoint intervals the sets of the real numbers $x \in \mathbb{R}$ satisfying:
i) $\big(x < 20$ & $x^2 < 100\big)$ or $x \notin (-\infty, -1]$; ii) $x < 20$ & $\big(x^2 < 100$ or $x \notin (-\infty, -1]\big)$.

(∗) Does the place of the parentheses matter?

7) Let A, B, X, Y, X', Y' be sets. Using the ZF axioms, prove: (i) $s(A) := A \cup \{A\}$ is a set called the successor (set) of A; (ii) $(X, Y) := \{\{X\}, \{X, Y\}\}$ is a set called the ordered pair with entries X, Y. Further, prove that the following hold:

a) $s(A) = s(B)$ iff $A = B$.
b) $(X, Y) = (X', Y')$ iff $X = X'$ and $Y = Y'$. Hence ordered pairs are not commutative(!)
c) $A \times B := \{(X, Y) \mid X \in A, Y \in B\}$ is a set, called the (Cartesion) product of A, B.

8) Using the definitions of $\cup$, $\cap$, $\setminus$, $\times$, prove the following:

a) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
More general: $(\cap_{i \in I} A_i) \cup C = \cap_{i \in I}(A_i \cup C)$ and $(\cup_{i \in I} A_i) \cap C = \cup_{i \in I}(A_i \cap C)$.
b) *de Morgan* laws: $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$, $C \setminus (A \cap B) = C \setminus A \cup C \setminus B$.
More general: $C \setminus (\cup_{i \in I} A_i) = \cap_{i \in I}(C \setminus A_i)$ and $C \setminus (\cap_{i \in I} A_i) = \cup_{i \in I}(C \setminus A_i)$.
c) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ and $(A \cap B) \times C = (A \times C) \cap (B \times C)$.
More general: $(\cup_{i \in I} A_i) \times C = \cup_{i \in I}(A_i \times C)$ and $(\cap_{i \in I} A_i) \times C = \cap_{i \in I}(A_i \times C)$.

Correspondences/maps. Recall the notion of a (functional) correspondence $R \subset A \times B$, and further: (i) a functional correspondence $R \subset A \times B$ defines a function $f_R : A \to B$; (ii) a function $f : A \to B$ defines a correspondence $R_f = \{(x, y) \mid x \in A, y = f(x)\} \subset A \times B$ called the graph of f. **Recall** that $f_{R_f} = f$ and $R_{f_R} = R$ (WHY).

9) Let $f : A \to B$, $g : B \to C$ be maps, and consider $g \circ f : A \to C$. Prove the following:

a) If f and g are injective (reps. surjective), then $g \circ f$ is injective (reps. surjective).
Is the converse assertion true, i.e., if $g \circ f$ is injective (reps. surjective), is it true that f and g are injective (reps. surjective)?
b) If f and g are bijective, then $g \circ f$ is bijective. Is the converse assertion true?

10) Using the ZFC system of axioms, prove/answer the following:

a) There is $f : A \to B$ injective iff there is $g : B \to A$ surjective.
b) $f : A \to B$ is injective iff $\exists\, g : B \to A$ surjective satisfying $g\big(f(x)\big) = x \;\forall\, x \in A$.

• **Recall**: Let $X \xrightarrow{h} Y$ be an arbitrary map. Then for $A \subset X$ and $B \subset Y$ one has:

1) $f(A) := \{y \in Y \mid \exists\, x \in A \text{ s.t. } f(x) = y\}$ is a subset of Y (WHY), the image of A under f.
2) $f^{-1}(B) := \{x \in X \mid f(x) \in B\}$ is a subset of X (WHY), the pre-image of B under f.

11) Prove/disprove that for all subsets $A', A'' \subset X$ and $B', B'' \subset Y$ one has:

a) $f(A' \cap A'') = f(A') \cap f(A'')$, respectively $f(A' \cup A'') = f(A') \cup f(A'')$.

b) $f^{-1}(B' \cap B'') = f^{-1}(B') \cap f^{-1}(B'')$, respectively $f^{-1}(B' \cup B'') = f^{-1}(B') \cup f^{-1}(B'')$.

(•) The same questions provided f is injective, resp. surjective, resp. bijective.

12) In the notation from Problem 11 above, define $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $A \mapsto f(A)$, and $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ by $B \mapsto f^{-1}(B)$. Prove/disprove/answer:

a) f_* is injective iff f is injective.

b) f^* is injective iff f is surjective.

c) What about surjectivity of f_*, respectively of f^*?

Cardinality of sets. Denote the cardinality of A by $|A|$, and say $|A| \leqslant |B| \overset{\text{Def}}{\longleftrightarrow} \exists\, f : A \to B$ injective. In particular, if $|A| \leqslant |B|$ and $B| \leqslant |C|$, then $|A| \leqslant |C|$ (WHY). Recall the *famous* & *difficult* **Theorem (Cantor–Schroeder–Bernstein)**:

$|A| \leqslant |B|$ and $|B| \leqslant |A|$ iff there exists $f : A \to B$ bijective.

Further, X is called finite of cardinality $|X| = n$, if either $X = \emptyset$ and then $|X| = 0$, or there is a bijection $f : \{1, \ldots, n\} \to X$, thus $X = \{x_i := f(i) \mid i = 1, \ldots, n\} = \{x_1, \ldots, x_n\}$.

13) Let X be a non-empty set and $f : X \to X$ be maps. Prove/disprove the following:

a) If X is finite iff every injective/surjective $f : X \to X$ is bijective.

b) If X is infinite iff there are injective/surjective $f : X \to X$ which are not bijective.

[**Hints** a): Proof by induction on $|X|$, etc. b): $\mathbb{N} \to \mathbb{N}$]

14) Let X be an arbitrary set, and $\mathcal{P}(X) := \{A \mid A \subseteq X\}$ be the power set of X. Prove:

a) If $|X| = n$ is finite, then $|\mathcal{P}(X)| = 2^n$.

b) One always has that $|X| < |\mathcal{P}(X)|$. Deduce from this that $|\mathbb{N}| < |\mathbb{R}|$.

[**Hint** to the second part of b): Define $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by $f(A) := \overline{a_0.a_1 \ldots a_n \ldots}$ for $A \subseteq N$, where $a_n = 1$ if $n \in N$, and $a_n = 0$ if $n \notin N$. Then $N \neq N'$ implies $x_N \neq x_{N'}$ (WHY), hence f is injective, etc...]

15) Let X, Y be finite sets, say $|X| = m$ and $|Y| = n$. Prove/disprove the following:

a) $|X \cup Y| + |X \cap Y| = |X| + |Y|$. What is the corresponding assertion for $|X \cup Y \cup Z|$?

b) $|X \times Y| = |X| \cdot |Y|$. What is the corresponding assertion for $|X \times Y \times Z|$?

[**Hint**: Proof by induction on m, n, etc.]

Proofs by contradiction / Direct proofs.

16) Prove that $\sqrt[3]{36}$ is not a rational number.

17) Prove that $e = \sum_n \frac{1}{n!}$ is not a rational number.

18) Let $X \subset \mathbb{Q}$, $X \neq \{0\}$ be closed w.r.t. subtraction and division, i.e., $\forall\, x, y \in X, y \neq 0$ one has: $x - y, x/y \in X$. Prove that $X = \mathbb{Q}$. What is the corresponding assertion for $X \subset \mathbb{Z}$?