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Math 314/514 Final Exam / Due: Tuesday, May 3, 2022, at 5 PM

Academic Integrity Statement:

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this Math 314/514 exam. [That means, among other things, that you are allowed to: (a) discuss the problems of the midterm with your colleagues, but not work out solutions together; (b) ask any member the Math Department about hints to the exam problems, but you must *first* mention to her/him that it goes about problems on a take home exam; (c) consult resources (books, Internet, etc.), but you must design/write down your own proofs.]

Name (printed): ________________________ Signature:

Note: There are ten problems on this exam.

Points: 10 points for each problem (partial and/or extra credit possible).

Grading: 55 < C−, C, C+ ⩽ 70 < B−, B, B+ ⩽ 85 < A−, A, A+

Procedures: Print out the two pages of the exam and staple them to your work. Write your name (printed) and sign the above Academic Integrity Statement. You must submit a **hard copy** of your exam.

• Recall: A complete proof must contain all the necessary explanations/steps, and in order to *disprove* an assertion you must give a counterexample showing that the assertion is not true.

1) Let V be an $\mathbb{R}$-vector space. Prove/disprove the following:

a) $\dim(V) \geqslant 3$ iff V has infinitely many vector subspaces W with $\dim(W) = 2$.

b) Let $V_1, V_2 \subset V$ be subspaces such that $V_1 \cup V_2 = V$. Then $V_1 = V$ or $V_2 = V$.

2) For $A \in F^{2\times 2}$ define $T_A : F^{2\times 3} \to F^{2\times 3}$ by $T_A(X) = AX$. Show that T_A is a linear transformation and prove/disprove/answer the following:

a) $\det(A) \neq 0$ if and only if the equation $T_A(X) = B$ in the unknown $X \in F^{2\times 3}$ has a solution in $F^{2\times 3}$ for each $B \in F^{2\times 3}$. Is in this case the solution X unique?

b) $\operatorname{im}(T_A) = F^{2\times 3}$ if and only if $\ker(T_A)$ consists of the zero matrix $\mathbf{0}_{2\times 3}$ only.

3) Let $A \in F^{n\times n}$ with $F = \mathbb{R}$ or $F = \mathbb{C}$ be given. Prove/disprove the following assertions:

a) $I_n + AA^*$ is invertible.

b) If A is self-adjoint, then $I_n + A$ is invertible.

c) If A is unitary, then $A + cI_n$ is invertible for c any real number satisfying $-1 < c < 1$.

4) Consider $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & a_{13} \\ a_{21} & a_{22} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & a_{32} & a_{33} \end{pmatrix} \in \mathbb{R}^{3\times 3}$ and $C = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ \frac{i}{2} & c_{32} & \frac{1}{2} \end{pmatrix} \in \mathbb{C}^{3\times 3}$.

Find values of the free entries, provided such values exist, such that A, respectively C, are:

a) unitary; b) self adjoint; c) orthogonal.

If in a specific case no such values exist, explain/justify your answer.

5) Discuss the existence and uniqueness of the solution the systems of equations below, where $a, b, b_1, \dots, b_m \in F$ are arbitrary in an arbitrary field F. (The answer *might depend on F*.)

a) $\begin{cases} ax + y + z = a \\ x + ay + z = 0 \\ x + y + az = 0 \\ x + y + z = b \end{cases}$ b) $A\boldsymbol{x} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$, where $A = (a_{ij})_{i,j} \in F^{m\times n}$, $a_{ij} = i + j$

6) For $A \in \mathbb{C}^{3\times 3}$, let $A = UN$ be a polar decomposition. Prove/disprove the following:

a) $\lambda \in F$ is eigenvalue of A if and only if $|\lambda|$ is eigenvalue of N.

b) $|\det(A)| = \det(N)$.

7) Let $B = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$, and $\boldsymbol{b} = \begin{pmatrix} a \\ 0 \\ 0 \\ a' \end{pmatrix}$ with $a, a' \in \mathbb{C}$ arbitrary. Find the following:

a) Representations of B of the form $B = U\Delta$ with U unitary and Δ upper triangular with non-negative diagonal entries.

b) Find $\boldsymbol{x} \in \mathbb{C}^{4\times 1}$ such that $B\boldsymbol{x}$ best approximates $\boldsymbol{b}$.

8) Let $\mathcal{C}(I, \mathbb{R})$ be the $\mathbb{R}$-vector space of continuous functions on $I = [-a, a]$ endowed with the standard inner product $(f \mid g) := \int_{-a}^{a} f(t)g(t)dt$, where $a = \frac{\pi}{2}$. Let $V \subset \mathcal{C}(I, \mathbb{R})$ be the subspace generated by $u_1 = 1$, $u_2 = \sin x$, $u_3 = \sin^2 x$. Answer the following:

a) Which of the functions $g_1(t) = t + 1$, $g_2(t) = \cos(2x)$, $g_3(t) = \sin(2x)$ belong to V?

b) Find an orthonormal basis of V.

c) Find the function $u(t) \in V$ which best approximates the polynomial $p(t) = t$ in the norm $\|\ \|$ defined by inner product $(\ |\ )$.

[**Hint** To a): Recall that $f = a_1u_1 + a_2u_2 + a_3u_3$ with $a_1, a_2, a_3 \in \mathbb{R}$ iff $f(x) = a_1u_1(x) + a_2u_2(x) + a_3u_3(x)$ for all $x \in I$, etc. To c): Let $E \subset W$ is a subspace, and $w \in W$ be given. Which $u \in E$ does best approximate $w \in W$?]

9) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation which maps the line $y = x$ onto the line $y = -x$, and the line $y = -x$ onto the line $y = x$. Find the matrix of T in the standard basis (e_1, e_2) of $\mathbb{R}^2$, in each of the following cases (each case considered separately):

a) $T(2, 3) = (-2, 1)$.

b) $T^3 = 4T$.

[**Hint:** $v \in \mathbb{R}^2$ is on the line $y = x$ iff $v = (a, a)$, and $w \in \mathbb{R}^2$ is on the line $y = -x$ iff $v = (-b, b)$, etc.]

10) Consider $A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, and $\boldsymbol{x} := (x_1, x_2, x_3)$. Answer the following:

a) Diagonalize the quadratic forms $q_A(\boldsymbol{x}) = \boldsymbol{x}A\boldsymbol{x}^\tau$ and $q_B(\boldsymbol{x}) = \boldsymbol{x}B\boldsymbol{x}^\tau$.

b) Are there invertible matrices $S, T \in \mathbb{R}^{3\times 3}$ such that $S^\tau A\, S = T^\tau B\, T$?

c) Which of the matrices $\pm A$, $\pm B$, $A \pm B$, $A^2 \pm B^2$ are (semi)positive definite?