Math 202 / Problem Set 9 (two pages)

Recall that for every \( x \in \mathbb{R} \) and \( r \in \mathbb{N}_{>0} \) the following hold (see HW #8, Problem 6):

a) If \( x \geq 0 \) there exists a **unique** \( y \in \mathbb{R} \), \( y \geq 0 \) such that \( y^r = x \).

b) If \( r \) is odd, for every \( x \in \mathbb{R} \) there is a **unique** \( y \in \mathbb{R} \) such that \( y^r = x \).

**Notation:** \( x^{\frac{1}{r}} := \sqrt[r]{x} := y \) is called the \( r \)-th root of \( x \).

**The \( r \)-th root function.**

Let \( r \in \mathbb{N}_{>0} \) be given. Define the \( r \)-th *root function* \( \sqrt[r]{\cdot} \) as follows:

I) If \( r \in \mathbb{N}_{>0} \) is even, define \( \sqrt[r]{\cdot} : [0, \infty) \to [0, \infty) \) by \( x \mapsto \sqrt[r]{x} = x^{\frac{1}{r}} \).

II) If \( r \in \mathbb{N} \) is odd, define \( \sqrt[r]{\cdot} : \mathbb{R} \to \mathbb{R} \) by \( x \mapsto \sqrt[r]{x} = x^{\frac{1}{r}} \).

1) Prove that the \( r \)-th root function \( \sqrt[r]{\cdot} \) is bijective and has the following properties:

a) \( \sqrt[r]{\cdot} \) is strictly increasing, i.e., \( x_1 < x_2 \) iff \( \sqrt[r]{x_1} < \sqrt[r]{x_2} \), that is, \( x_1^{\frac{1}{r}} < x_2^{\frac{1}{r}} \).

b) \( \sqrt[r]{\cdot} \) is multiplicative, i.e., \( \sqrt[r]{x_1 x_2} = \sqrt[r]{x_1} \sqrt[r]{x_2} \), that is, \( (x_1 x_2)^{\frac{1}{r}} = x_1^{\frac{1}{r}} x_2^{\frac{1}{r}} \).

c) \( \sqrt[r]{\cdot} \circ \sqrt[r]{\cdot} = \sqrt[r]{\cdot} \circ \sqrt[r]{\cdot} \), or \( (x^{\frac{1}{r}})^{\frac{1}{r}} = x^{\frac{1}{r \cdot r}} = (x^{\frac{1}{r}})^{\frac{1}{r}} \) (provided everything is defined).

**The power-\( \alpha \) function** \( f_\alpha : (0, \infty) \to (0, \infty) \), \( x \mapsto x^\alpha \)

2) Recalling that \( x^{-a} := \frac{1}{x^a} \) for \( x \in \mathbb{R} \), \( x \neq 0 \) and \( a \in \mathbb{Z} \), prove the following:

a) \( (x^{\frac{1}{r}})^a = (\sqrt[r]{x})^a = \sqrt[r]{x^a} = (x^a)^{\frac{1}{r}} \);  
b) \( (x^{\frac{1}{r}})^{-a} = (\sqrt[r]{x})^{-a} = \frac{1}{\sqrt[r]{x^a}} = \frac{1}{(x^a)^{\frac{1}{r}}} \).

3) Let \( \alpha = \frac{a}{r} \in \mathbb{Q} \), \( a \in \mathbb{Z} \). Define \( f_\alpha : (0, \infty) \to (0, \infty) \) by \( x \mapsto x^\alpha := (x^{\frac{1}{r}})^a = (x^a)^{\frac{1}{r}} \).

Prove that \( f_\alpha : (0, \infty) \to (0, \infty) \) satisfies:

a) \( f_\alpha \) is strictly increasing for \( \alpha > 0 \), strictly decreasing if \( \alpha < 0 \), and \( f_\alpha = 1 \) for \( \alpha = 0 \).

b) \( f_\alpha(x_1 x_2) = (x_1 x_2)^\alpha = x_1^\alpha x_2^\alpha = f_\alpha(x_1)f_\alpha(x_2) \), i.e. \( f_\alpha \) is multiplicative.

c) \( f_\alpha \cdot f_\beta)(x) = x^{\alpha + \beta} = f_{\alpha + \beta}(x) \), hence \( f_\alpha \cdot f_\beta = f_{\alpha + \beta} \) as maps.

d) \( f_\alpha \left( f_\beta(x) \right) = (x^\alpha)^\beta = x^{\alpha \beta} = (x^\beta)^\alpha = f_\beta \left( f_\alpha(x) \right) \), i.e., \( f_\alpha \circ f_\beta = f_{\alpha \beta} = f_\beta \circ f_\alpha \).

4) Let \( u \in \mathbb{R} \) and \( x \in \mathbb{R}, x > 0 \) be given. **Define** \( x^u \) by proving the following:

a) Let \( (a_n)_n \subseteq \mathbb{S}(\mathbb{Q}) \) satisfy \( a_n \to u \). Then \( (x^{a_n})_n \) is convergent in \( \mathbb{R} \).

b) If \( a_n - b_n \to 0 \) in \( \mathbb{Q} \), then \( x^{a_n} - x^{b_n} \to 0 \). **Hence** \( x^u := \lim_{n \to \infty} x^{a_n} \) is well defined.

**[Hint:]** If \( (a_n)_n \) is monotone, so is \( (x^{a_n})_n \) **(Why)**. Every \( a_n \to 0 \) has a subsequence \( (a_{n_k})_k \) with \( |a_{n_k}| < \frac{1}{k} \) **(Why)**. Since \( x^{\frac{1}{k}} \to 1 \) **(Why)**, conclude that \( x^{a_n} \to 1 \). If \( a_n \not\to u, b_n \not\to u \), get \( \lim_{n \to \infty} x^{a_n} = x^u = \lim_{n \to \infty} x^{b_n} \) **(Why)**, etc.

5) Let \( \alpha \in \mathbb{R} \). Define \( f_\alpha : (0, \infty) \to (0, \infty) \) by \( x \mapsto x^\alpha \) as defined above. Prove the following:

a) \( f_\alpha \) is strictly increasing for \( \alpha > 0 \), strictly decreasing if \( \alpha < 0 \), and \( f_\alpha = 1 \) for \( \alpha = 0 \).

b) One has \( f_\alpha(x_1 x_2) = (x_1 x_2)^\alpha = x_1^\alpha x_2^\alpha = f_\alpha(x_1)f_\alpha(x_2) \), i.e. \( f_\alpha \) is multiplicative.

c) \( f_\alpha \cdot f_\beta(x) = x^{\alpha + \beta} = f_{\alpha + \beta}(x) \), hence \( f_\alpha \cdot f_\beta = f_{\alpha + \beta} \) as maps.

d) \( f_\alpha \left( f_\beta(x) \right) = (x^\alpha)^\beta = x^{\alpha \beta} = (x^\beta)^\alpha = f_\beta \left( f_\alpha(x) \right) \), i.e., \( f_\alpha \circ f_\beta = f_{\alpha \beta} = f_\beta \circ f_\alpha \).
Conclude: Let $\alpha \beta = 1$. Then $f_\alpha \circ f_\beta(x) = x = f_\beta \circ f_\alpha(x)$ for all $x \in \mathbb{R}$ (WHY). Therefore, $f_\alpha$ is bijective for all $\alpha \neq 0$, and moreover, its inverse map is $f_\alpha^{-1} = f_\alpha^R$ (WHY).

The exponential function $\exp_\alpha : \mathbb{R} \to (0, \infty)$

Let $a > 0$ be a fixed positive real number. Define $\exp_\alpha : \mathbb{R} \to (0, \infty)$ by $x \mapsto a^x$.

Note that if $a = 1$, then $a^x = 1 \forall x \in \mathbb{R}$ (WHY). Hence one considers only the cases $a > 0, a \neq 1$.

6) Let $a \in \mathbb{R}_{>0}, a \neq 1$. Prove the following:
   a) If $1 < a$, then $\exp_a$ is strictly increasing, i.e., $x_1 < x_2$ in $\mathbb{R}$, then $a^{x_1} < a^{x_2}$.
   b) If $0 < a < 1$, then $\exp_a$ is strictly decreasing, i.e., $x_1 < x_2$ in $\mathbb{R}$, then $a^{x_1} > a^{x_2}$.
   c) $\exp_a(x_1 + x_2) = a^{x_1}a^{x_2} = \exp_a(x_1)\exp_a(x_2)$ for all $x_1, x_2 \in \mathbb{R}$.

7) In the above notations, prove the following facts about $f_\alpha$ and $\exp_\alpha$ as defined above:
   a) If $x_n \to x$ in $(0, \infty)$, then $f_\alpha(x_n) \to f_\alpha(x)$ in $(0, \infty)$.
   b) If $x_n \to x$ in $\mathbb{R}$, then $\exp_a(x_n) \to \exp_a(x)$ in $(0, \infty)$.

[Hint For $x_n \to x$, choose $u_n, v_n \to x$ with $u_n, v_n \in \mathbb{Q}, u_n \leq x_n \leq v_n$. Conclude by using that $f_\alpha$ and $\exp_a$ are monotone, and applying the Squeeze Thm, etc. . . .]

8) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-zero map which is compatible with addition & multiplication, i.e., $\forall x, y \in \mathbb{R}$ one has: $\varphi(x+y) = \varphi(x) + \varphi(y), \varphi(xy) = \varphi(x)\varphi(y)$ and $\exists x \in \mathbb{R}$ s.t. $\varphi(x) \neq 0$.

Prove the following surprising and interesting fact: $\varphi(x) = x$ for all $x \in \mathbb{R}$.

[Hint: Go through the following steps:
   a) $\varphi(0) = 0$ and $\varphi(1) = 1$. Further, $\varphi(-x) = -\varphi(x), \varphi(1/x) = 1/\varphi(x)$ if $x \neq 0$.
   b) $\varphi(a) = a$ for all $a \in \mathbb{Z}$, hence $\varphi(a/r) = a/r$ for all $a/r \in \mathbb{Q}$.
   c) $\varphi$ is compatible with the ordering $\leq$ of $\mathbb{R}$, i.e., $x \leq y \iff \varphi(x) \leq \varphi(y)$.
   d) Finally prove that $\varphi(x) = x$ for all $x \in \mathbb{R}$.

To a): $0 + 0 = 0 \Rightarrow \varphi(0) + \varphi(0) = \varphi(0)$, etc; $x \cdot 1 = x \Rightarrow \varphi(x)\varphi(1) = \varphi(x)$, thus $\varphi(x) \neq 0 \Rightarrow \varphi(1) \neq 0$, etc; $1 \cdot 1 = 1 \Rightarrow \varphi(1) \cdot \varphi(1) = \varphi(1)$, etc; $x + (-x) = 0 \Rightarrow \varphi(-x) = -\varphi(x)$ (WHY), $x(1/x) = 1 \Rightarrow \varphi(1) = x/\varphi(x)$ (WHY), etc.

To b): Since $\varphi(1) = 1$, prove by induction that $\varphi(n) = n$, hence $\varphi(-n) = -n$ (WHY), and $\varphi(\pm \frac{m}{n}) = \pm \frac{m}{n}$ $\forall m, k \in \mathbb{N}_{>0}$ (WHY).

To c): Use that $x > 0$ iff $x = y^2$ for some $y \in \mathbb{R}$, etc.

To d): For $x \in \mathbb{R}$, let $(x_n)_n, (y_n)_n \in S(\mathbb{Q})$ with $x_n \leq x \leq y_n \forall n$ and $x_n \to x \leftarrow y_n$. Then $x_n \leq \varphi(x) \leq y_n$ for all $n$ (WHY), etc.]

Have fun!

Which of the following series converges in $\mathbb{R}$, and if so, what is the sign of the represented number:

a) The harmonic series: $\sum_{n>0} \frac{1}{n}$; Leibniz alternating series: $\sum_{n>0} \frac{(-1)^n}{n}$

b) $\zeta(2) = \sum_{n>0} \frac{1}{n^2}$.

c) $\sin(1) = \sum_n \frac{(-1)^n}{(2n+1)!}; \cos(1) = \sum_n \frac{(-1)^n}{(2n)!}$.

d) $(\frac{3}{2})^\alpha = \sum_n \frac{\alpha}{n!}$. More general, $(1+z)^\alpha = \sum_n \frac{\alpha}{n!} z^n$ for $z \in \mathbb{R}$.

[Here, by definitions, $\binom{\alpha}{0} \overset{\def}{=} 1$, and $\binom{\alpha}{n} \overset{\def}{=} \frac{\alpha\ldots(\alpha-n+1)}{n!}$ for $n > 0$.]