Due: Thursday, Nov 11, 2021

Math 202 / Problem Set 9 (two pages)

Recall that for every $x \in \mathbb{R}$ and $r \in \mathbb{N}_{>0}$ the following hold (see HW # 8, Problem 6):

a) If $x \geqslant 0$ there exists a ***unique*** $y \in \mathbb{R}$, $y \geqslant 0$ such that $y^r = x$.
b) If r is odd, for every $x \in \mathbb{R}$ there is a ***unique*** $y \in \mathbb{R}$ such that $y^r = x$.

Notation: $x^{\frac{1}{r}} :\overset{\text{def}}{=} \sqrt[r]{x} :\overset{\text{def}}{=} y$ is called the r^{th} *root of* x.

The r^{th}-root function.

Let $r \in \mathbb{N}_{>0}$ be given. Define the r^{th}-*root function* $\sqrt[r]{\ }$ as follows:

I) *If* $r \in \mathbb{N}_{>0}$ *is even, define* $\sqrt[r]{\ } : [0, \infty) \to [0, \infty)$ *by* $x \mapsto \sqrt[r]{x} = x^{\frac{1}{r}}$.
II) *If* $r \in \mathbb{N}$ *is odd, define* $\sqrt[r]{\ } : \mathbb{R} \to \mathbb{R}$ *by* $x \mapsto \sqrt[r]{x} = x^{\frac{1}{r}}$.

1) Prove that the r^{th}-root function $\sqrt[r]{\ }$ is bijective and has the following properties:

a) $\sqrt[r]{\ }$ is strictly increasing, i.e., $x_1 < x_2$ iff $\sqrt[r]{x_1} < \sqrt[r]{x_2}$, that is, $x_1^{\frac{1}{r}} < x_2^{\frac{1}{r}}$.
b) $\sqrt[r]{\ }$ is multiplicative, i.e., $\sqrt[r]{x_1 x_2} = \sqrt[r]{x_1}\,\sqrt[r]{x_2}$, that is, $(x_1 x_2)^{\frac{1}{r}} = x_1^{\frac{1}{r}} x_2^{\frac{1}{r}}$.
c) $\sqrt[r]{\ } \circ \sqrt[s]{\ } = \sqrt[rs]{\ } = \sqrt[s]{\ } \circ \sqrt[r]{\ }$, or $(x^{\frac{1}{s}})^{\frac{1}{r}} = x^{\frac{1}{rs}} = (x^{\frac{1}{r}})^{\frac{1}{s}}$ (provided everything is defined).

The power-α function $f_\alpha : (0, \infty) \to (0, \infty)$, $x \mapsto x^\alpha$

2) Recalling that $x^{-a} :\overset{\text{def}}{=} \frac{1}{x^a}$ for $x \in \mathbb{R}$, $x \neq 0$ and $a \in \mathbb{Z}$, prove the following:

a) $(x^{\frac{1}{r}})^a = (\sqrt[r]{x})^a = \sqrt[r]{x^a} = (x^a)^{\frac{1}{r}}$; b) $(x^{\frac{1}{r}})^{-a} = (\sqrt[r]{x})^{-a} = \frac{1}{\sqrt[r]{x^a}} = \frac{1}{(x^a)^{\frac{1}{r}}}$.

3) Let $\alpha = \frac{a}{r} \in \mathbb{Q}$, $a \in \mathbb{Z}$. Define $f_\alpha : (0, \infty) \to (0, \infty)$ by $x \mapsto x^\alpha :\overset{\text{def}}{=} (x^{\frac{1}{r}})^a = (x^a)^{\frac{1}{r}}$. Prove that $f_\alpha : (0, \infty) \to (0, \infty)$ satisfies:

a) f_α is strictly increasing for $\alpha > 0$, strictly decreasing if $\alpha < 0$, and $f_\alpha = 1$ for $\alpha = 0$.
b) $f_\alpha(x_1 x_2) = (x_1 x_2)^\alpha = x_1^\alpha x_2^\alpha = f_\alpha(x_1) f_\alpha(x_2)$, i.e. f_α is multiplicative.
c) $(f_\alpha \cdot f_\beta)(x) = x^{\alpha+\beta} = f_{\alpha+\beta}(x)$, hence $f_\alpha \cdot f_\beta = f_{\alpha+\beta}$ as maps.
d) $f_\alpha\big(f_\beta(x)\big) = (x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha = f_\beta\big(f_\alpha(x)\big)$, i.e., $f_\alpha \circ f_\beta = f_{\alpha\beta} = f_\beta \circ f_\alpha$.

4) Let $u \in \mathbb{R}$ and $x \in \mathbb{R}$, $x > 0$ be given. **Define** x^u by proving the following:

a) Let $(a_n)_n \in \mathcal{S}(\mathbb{Q})$ satisfy $a_n \to u$. Then $(x^{a_n})_n$ is convergent in $\mathbb{R}$.
b) If $a_n - b_n \to 0$ in $\mathbb{Q}$, then $x^{a_n} - x^{b_n} \to 0$. **Hence** $x^u := \lim_{n\to\infty} x^{a_n}$ is well defined.

[**Hint**: If $(a_n)_n$ is monotone, so is $(x^{a_n})_n$ (WHY). Every $a_n \to 0$ has a subsequence $(a_{n_k})_k$ with $|a_{n_k}| < \frac{1}{k}$ (WHY). Since $x^{\frac{1}{k}} \to 1$ (WHY), conclude that $x^{a_n} \to 1$. If $a_n \nearrow u$, $b_n \searrow u$, get $\lim_{n\to\infty} x^{a_n} =: x^u := \lim_{n\to\infty} x^{b_n}$ (WHY), etc.

5) Let $\alpha \in \mathbb{R}$. Define $f_\alpha : (0, \infty) \to (0, \infty)$ by $x \mapsto x^\alpha$ as defined above. Prove the following:

a) f_α is strictly increasing for $\alpha > 0$, strictly decreasing if $\alpha < 0$, and $f_\alpha = 1$ for $\alpha = 0$.
b) One has $f_\alpha(x_1 x_2) = (x_1 x_2)^\alpha = x_1^\alpha x_2^\alpha = f_\alpha(x_1) f_\alpha(x_2)$, i.e. f_α is multiplicative.
c) $(f_\alpha \cdot f_\beta)(x) = x^{\alpha+\beta} = f_{\alpha+\beta}(x)$, hence $f_\alpha \cdot f_\beta = f_{\alpha+\beta}$ as maps.
d) $f_\alpha\big(f_\beta(x)\big) = (x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha = f_\beta\big(f_\alpha(x)\big)$, i.e., $f_\alpha \circ f_\beta = f_{\alpha\beta} = f_\beta \circ f_\alpha$.

Conclude: Let $\alpha\beta = 1$. Then $f_\alpha \circ f_\beta(x) = x = f_\beta \circ f_\alpha(x)$ for all $x \in \mathbb{R}$ (WHY). Therefore, f_α is bijective for all $\alpha \neq 0$, and moreover, its inverse map is $f_\alpha^{-1} = f_{\frac{1}{\alpha}}$ (WHY).

The exponential function $\exp_a : \mathbb{R} \to (0, \infty)$

Let $a > 0$ be a fixed positive real number. Define $\exp_a : \mathbb{R} \to (0, \infty)$ by $x \mapsto a^x$.
Note that if $a = 1$, then $a^x = 1 \,\forall\, x \in \mathbb{R}$ (WHY). Hence one considers only the cases $a > 0, a \neq 1$.

6) Let $a \in \mathbb{R}_{>0}$, $a \neq 1$. Prove the following:

a) If $1 < a$, then $\exp_a$ is strictly increasing, i.e., $x_1 < x_2$ in $\mathbb{R}$, then $a^{x_1} < a^{x_2}$.

b) If $0 < a < 1$, then $\exp_a$ is strictly decreasing, i.e., $x_1 < x_2$ in $\mathbb{R}$, then $a^{x_1} > a^{x_2}$.

c) $\exp_a(x_1 + x_2) = a^{x_1+x_2} = a^{x_1}a^{x_2} = \exp_a(x_1)\exp_a(x_2)$ for all $x_1, x_2 \in \mathbb{R}$.

7) In the above notations, prove the following facts about f_α and $\exp_a$ as defined above:

a) If $x_n \to x$ in $(0, \infty)$, then $f_\alpha(x_n) \to f_\alpha(x)$ in $(0, \infty)$.

b) If $x_n \to x$ in $\mathbb{R}$, then $\exp_a(x_n) \to \exp_a(x)$ in $(0, \infty)$.

[**Hint** For $x_n \to x$, choose $u_n, v_n \to x$ with $u_n, v_n \in \mathbb{Q}$, $u_n \leqslant x_n \leqslant v_n$. Conclude by using that f_α and $\exp_a$ are monotone, and applying the Squeeze Thm, etc. ...]

8) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-zero map which is compatible with addition & multiplication, i.e., $\forall\, x, y \in \mathbb{R}$ one has: $\varphi(x+y) = \varphi(x)+\varphi(y)$, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\exists\, x \in \mathbb{R}$ s.t. $\varphi(x) \neq 0$.
Prove the following *surprising and interesting* fact: $\varphi(x) = x$ for all $x \in \mathbb{R}$.

[**Hint:** Go through the following steps:

a) $\varphi(0) = 0$ and $\varphi(1) = 1$. Further, $\varphi(-x) = -\varphi(x)$, $\varphi(1/x) = 1/\varphi(x)$ if $x \neq 0$.

b) $\varphi(a) = a$ for all $a \in \mathbb{Z}$, hence $\varphi(a/r) = a/r$ for all $a/r \in \mathbb{Q}$.

c) φ is compatible with the ordering $\leqslant$ of $\mathbb{R}$, i.e., $x \leqslant y$ iff $\varphi(x) \leqslant \varphi(y)$.

d) Finally *prove that* $\varphi(x) = x$ *for all* $x \in \mathbb{R}$.

To a): $0 + 0 = 0 \Rightarrow \varphi(0) + \varphi(0) = \varphi(0)$, etc; $x \cdot 1 = x \Rightarrow \varphi(x)\varphi(1) = \varphi(x)$, thus $\varphi(x) \neq 0 \Rightarrow \varphi(1) \neq 0$, etc;
$1 \cdot 1 = 1 \Rightarrow \varphi(1) \cdot \varphi(1) = \varphi(1)$, etc; $x + (-x) = 0 \Rightarrow \varphi(-x) = -\varphi(x)$ (WHY), $x(1/x) = 1 \Rightarrow \varphi(\frac{1}{x}) = \frac{1}{\varphi(x)}$ (WHY), etc.
To b): Since $\varphi(1) = 1$, prove by induction that $\varphi(n) = n$, hence $\varphi(-n) = -n$ (WHY), and $\varphi(\pm\frac{m}{k}) = \pm\frac{m}{k}\ \forall\, m, k \in \mathbb{N}_{>0}$ (WHY).
To c): Use that $x > 0$ iff $x = y^2$ for some $y \in \mathbb{R}$, etc.
To d): For $x \in \mathbb{R}$, let $(x_n)_n, (y_n)_n \in \mathcal{S}(\mathbb{Q})$ with $x_n \leqslant x \leqslant y_n\ \forall\, n$ and $x_n \to x \leftarrow y_n$. Then $x_n \leqslant \varphi(x) \leqslant y_n$ for all n (WHY), etc.]

Have fun!

Which of the following series converges in $\mathbb{R}$, and if so, what is the sign of the represented number:

a) The harmonic series: $\sum_{n>0} \frac{1}{n}$; Leibniz alternating series: $\sum_{n>0} \frac{(-1)^n}{n}$

b) $\zeta(2) = \sum_{n>0} \frac{1}{n^2}$;

c) $\sin(1) = \sum_n \frac{(-1)^n}{(2n+1)!}$; $\cos(1) = \sum_n \frac{(-1)^n}{(2n)!}$.

d) $(\frac{3}{2})^\alpha = \sum_n \binom{\alpha}{n}\frac{1}{2^n}$. More general, $(1 + z)^\alpha = \sum_n \binom{\alpha}{n} z^n$ for $z \in \mathbb{R}$.

[**Here**, by definitions, $\binom{\alpha}{0} :\overset{\text{def}}{=} 1$, and $\binom{\alpha}{n} :\overset{\text{def}}{=} \frac{\alpha \ldots (\alpha-n+1)}{n!}$ for $n > 0$.]