Due: Friday, Nov 6, 2021

Math 202 / Problem Set 8 (two pages)

Sequences (continued)
Recall that $F, +, \cdot \leqslant$ is a totally ordered field, and that $\mathcal{S}_{\rm c}(F) \subset \mathcal{S}_{\rm C}(F) \subset \mathcal{S}_{\rm b}(F)$.
Terminology: $(a_n)_n \in \mathcal{S}(F)$ is called divergent to ∞, respectively $-\infty$, if $(a_n)_n$ satisfies: $\forall\, \eta_F$ one has $\eta < a_n$, respectively $a_n < -\eta$, for $n \gg 0$. **Notation**. $a_n \to \infty$, or $\lim\limits_{n\to\infty} a_n = \infty$.

1) Let $(a_n)_n \in \mathcal{S}(F)$ be given. Prove/disprove/answer the following:

a) $(a_n)_n$ is bounded iff $(a_n^2)_n$ is bounded.

b) Using quantifiers, write the assertion: "*$(a_n)_n$ is Cauchy, but not convergent.*"

c) $(a_n)_n$ is convergent to $a \in F$ iff $(a_n^3)_n$ is convergent to $a^3 \in F$.

2) True or false (**justify** your answer, that is, prove/disprove the assertion):

a) Suppose that $a_n \neq 0_F\ \forall\, n \in \mathbb{N}$. Then $a_n \to 0_F$ iff $a^2 \to 0_F$ iff $\frac{1}{a_n^2} \to \infty$ iff $\frac{1}{a_n} \to \infty$.

b) Let $f(t) = a_0 + a_1 t + \dots + a_n t^n \in F[t]$ be a polynomial. For $(x_n)_n \in \mathcal{S}(F)$, set $y_n := f(x_n)$, and consider $(y_n)_n \in \mathcal{S}(F)$. Prove/disprove:
(i) If $x_n \to x \in F$, then $y_n \to f(x) \in F$. (ii) If $(x_n)_n$ is Cauchy, so is $(y_n)_n$.

Completions / Field of real numbers $\mathbb{R}$

- For a totally ordered field $F, +, \cdot, \leqslant$ let $\widehat{F} := \mathcal{C}(F)/\!\sim$ be its Cauchy completion.
- For $(a_n)_n$ Cauchy in F, we denote by $\widehat{a} := (a_n)_n/\!\sim$ the corresponding element of $\widehat{F}$.

Recall: $\imath_{\widehat{F}} : F \to \widehat{F}$ by $a \mapsto (a)_n/\!\sim$, is compatible with $+$, $\cdot$ and $\leqslant$, and we identify $a \in F$ with $\imath_{\widehat{F}}(a)$, thus identify F with (the subfield) $\imath_{\widehat{F}}(F) \subset \widehat{F}$, and write $F \hookrightarrow \widehat{F}$.
Study & complete all details of the proofs of the assertions (from the class):

a) For $(a_n)_n \in \mathcal{S}_{\rm C}(F)$, and $\widehat{\boldsymbol{a}} = (a_n)_n/\!\sim$ in $\widehat{F}$, then $a_n \to \widehat{\boldsymbol{a}}$ in $\widehat{F}$.

b) F is dense in $\widehat{F}$, i.e., if $x < y$ in $\widehat{F}$, then $\exists\, a \in F$ s.t. $x < a < y$.

c) For $(x_n)_n$ Cauchy in $\widehat{F}$, there is $(a_n)_n$ Cauchy in F s.t. $(x_n)_n \sim (a_n)_n$.
Hence if $x = (a_n)_n/\!\sim$ in $\widehat{F}$, then $a_n \to x$ in $\widehat{F}$ (WHY), one finally has $x_n \to x$ (WHY).

Conclude:

(●) *F is dense in $\widehat{F}$, every Cauchy sequence in $\widehat{F}$ is convergent, thus $\widehat{\widehat{F}} = \widehat{F}$.*

(∗) *$\mathbb{Q}$ is dense $\mathbb{R} = \widehat{\mathbb{Q}}$, every Cauchy sequence in $\mathbb{R}$ is convergent, thus $\widehat{\mathbb{R}} = \mathbb{R}$.*

d) Let $F' \subset F$ be a dense subfield. Then for every Cauchy sequence $(a_n)_n \in \mathcal{S}_{\rm C}(F)$ there exists a Cauchy sequence $(a'_n)_n$ over F' such that $(a'_n)_n \sim (a_n)_n$ in $\mathcal{S}_{\rm C}(F)$.

e) Let $F' \subset F$ be a dense subfield. Then $\widehat{F'} = \widehat{F}$.

f) Hence if $\mathbb{Q} \subset F$ is dense, one has $\widehat{F} = \widehat{\mathbb{Q}} = \mathbb{R}$.

Conclude:

Theorem. *$\mathbb{R}$ is the unique totally ordered complete field in which $\mathbb{Q}$ is dense.*

3) Check all the details in the proofs of the following assertions from the class:

a) For a totally ordered field $F, +, \cdot, \leqslant$ the following are equivalent:

(i) F satisfies the *Archimedean Axiom*: $\forall\, \epsilon > 0_F$, $\forall\, x \in F\ \exists\, n \in \mathbb{N}$ s.t. $x < n\epsilon$.

(ii) $\frac{1}{n} 1_F \to 0_F$. (ii)′ $n \to \infty$.

(iii) For all $x < y$ in F $\exists\, a \in \mathbb{Q}$ s.t. $x < a1_F < y$, i.e., $\mathbb{Q}1_F$ is dense in F.

b) For a totally ordered field $F, +, \cdot, \leqslant$ the following are equivalent:

(i) For all $X \subset F$ non-empty and bounded above, $\sup(X)$ exists in F.

(ii) For all $Y \subset F$ non-empty and bounded below, $\inf(Y)$ exists in F.

(iii) For all $Z \subset F$ non-empty and bounded, $\sup(Z)$ and $\inf(Z)$ exist in F.

General facts about $\mathbb{R}$

4) Prove/disprove/answer the following:

a) Every real number $x \in \mathbb{R}$ has a *decimal representation*, i.e., for every $x \geqslant 0$ there is $(a_n)_n$ with $a_n \in \mathbb{N}$ and $0 \leqslant a_n < 10$ for $n > 0$, such that $\sum_{i=0}^{n} \frac{a_i}{10^i}$ is convergent and represents x. What is to do if $x < 0$?

•) Is the sequence $(a_n)_n$ unique for all / some real numbers $x \in \mathbb{R}$?

b) Show that the same holds with 10 replaced by any other basis $k > 1$, e.g. by $k = 2$ (leading to the so called *binary representation* of numbers — used in computers).

[**Hint**: Step 1. The set of rational numbers of the form $\{\frac{a}{k^r} \mid a \in \mathbb{Z}, r \in \mathbb{N}\}$ is closed w.r.t. $+, \cdot$ and it is dense in $\mathbb{Q}$.
Step 2. Every $a \in \mathbb{Z}$ can be written in basis k, say $n = r_0 + r_1 k + \cdots + r_m k^m$ (WHY).
Step 3. What is the form of $\frac{n}{k^r}$?, etc...

5) Prove/answer the following basic facts about real numbers:

a) For $x \in \mathbb{R}$ one has: $(x^n)_n \in \mathcal{S}_c(\mathbb{R})$ iff $-1 < x \leqslant 1$. Limits?

b) If $x \in \mathbb{R}$ and $x > 1$, then $x^n \to \infty$. What happens if $x \leqslant -1$?

The n^{th} root of a real number

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we say that $y \in \mathbb{R}$ is an n^{th} root of x, if $y^n = x$.

6) Prove the following facts about the n^{th} roots of real numbers $x \in \mathbb{R}$:

a) If $n = 2k$ is positive even and $x < 0$, *there is no n^{th} root of x.*

b) If $n > 0$ and $x \geqslant 0$, there is a *unique* n^{th} root $y \geqslant 0$ of x.

c) If n is odd, every $x \in \mathbb{R}$ has a unique n^{th} root y, and $-y$ is the n^{th} root of $-x$.

Notation. The n^{th} root y of x in the cases b), c), above is denoted $y = \sqrt[n]{x}$.

[**Hint**: For $x \geqslant 0$, let $X_x := \{a \in \mathbb{R} \mid 0 \leqslant a,\ a^n \leqslant x\}$. Then X_x is bounded (WHY), hence $y := \sup(X_x)$ exists and $y^n \leqslant x$ (WHY). By contradiction, suppose that $y^n < x$, hence $x = y^n + \epsilon_0$ for some $\epsilon > 0$. Exploration: For $0 < \epsilon < y$, one has: $(y+\epsilon)^n = y^n + \epsilon \sum_{i=1}^{n} \binom{n}{i} y^{n-i} \epsilon^{i-1} < y^n + \epsilon(n! y^{n-1})$ (WHY), hence $(y+\epsilon)^n < y^n + \epsilon(n! y^{n-1})$. Then choosing $\epsilon < \epsilon_0/(n! y^{n-1})$, and $y' := y + \epsilon$, get: $y'^n = (y+\epsilon)^n < x$ (WHY), hence $y' \in X_x$. OTOH, $y < y'$, hence $y < \sup(X_x)$ (WHY), **contradiction!**]

7) Let $k \in \mathbb{N}_{>0}$ be fixed. Prove/disprove/answer the following:

a) $\sqrt[n]{n^k} \to 1$ for all fixed $k \in \mathbb{N}$.

b) Let $(x_n)_n \in \mathcal{S}(\mathbb{R})$ satisfy $x_n \geqslant 0$ and $\frac{1}{n^k} \leqslant x_n^n \leqslant n^k$ for all $n \in \mathbb{N}_{>0}$. Then $x_n \to 1$.