Due: Fr, Oct 29, 2021

Math 202 / Problem Set 7 (two pages)

Make sure that you know/review/prove in all detail the following facts:

- The absolute value $|\ |: R \to R_{\geqslant 0}$ on a totally ordered domain $R, +, \cdot, \leqslant$ satisfies:
  - For $\epsilon > 0_R$ one has: (i) $|x| < \epsilon$ iff $-\epsilon < x < \epsilon$; (ii) $|x| \leqslant \epsilon$ iff $-\epsilon \leqslant x \leqslant \epsilon$.
  - (i) $|x+y| \leqslant |x| + |y|$; (ii) $|x \cdot y| = |x| \cdot |y|$; (iii) $\big||x| - |y|\big| = \max\big(|x|, |y|\big) - \min\big(|x|, |y|\big)$
- For $R, +, \cdot, \leqslant$ totally ordered domain, the map $\imath_{\mathbb{Z}} : \mathbb{Z} \to R$, $\imath_{\mathbb{Z}}(\pm n) = n(\pm 1_R)$, is injective and compatible with $+$, $\cdot$ and the ordering.
- For $F, +, \cdot, \leqslant$ totally ordered field, the map $\imath_{\mathbb{Q}} : \mathbb{Z} \to F$, $\imath(\frac{a}{n}) :\overset{\text{def}}{=} \frac{a}{n} 1_F :\overset{\text{def}}{=} (a1_F) \cdot (n1_F)^{-1}$ for $n > 0$ is well defined and compatible with $+$, $\cdot$ and ordering.

Sequences. Recall: "$\exists\, N \in \mathbb{N}$ s.t. $\forall\, n > N$ one has …" is the same as "… for $n \gg 0$"

- Recall the notations $\mathcal{S}(X) := \mathrm{Maps}(\mathbb{N}, X)$, and $\mathcal{S}_{\mathrm{c}}(F), \mathcal{S}_{\mathrm{C}}(F), \mathcal{S}_{\mathrm{b}}(F) \subset \mathcal{S}(F)$ are the sets of convergent/Cauchy/bounded sequences with values in F. Recall the definitions:
  - $a_n \to a \overset{\text{def}}{\Longleftrightarrow} \forall\, \epsilon > 0_F\ \exists\, N \in \mathbb{N}$ s.t. for $n > N$ one has $|a_n - a| < \epsilon$.
    $\overset{\text{def}}{\Longleftrightarrow} \forall\, \epsilon > 0_F$ one has $|a_n - a| < \epsilon$ for $n \gg 0$.
  - $(a_n)_n$ is Cauchy $\overset{\text{def}}{\Longleftrightarrow} \forall\, \epsilon > 0_F\ \exists\, N \in \mathbb{N}$ s.t. for $n, m > N$ one has $|x_n - x_m| < \epsilon$.
    $\overset{\text{def}}{\Longleftrightarrow} \forall\, \epsilon > 0_F$ one has $|a_m - a_n| < \epsilon$ for $m, n \gg 0$.
  - $(a_n)_n$ is bounded $\overset{\text{def}}{\Longleftrightarrow} \exists\, \epsilon_0 > 0_F$ s.t. $|x_n| < \epsilon_0\ \forall\, n \in \mathbb{N}$.

1) Let $\prec$ denote $\leqslant$ or $<$, and $\succ$ denote $\geqslant$ or $>$. Prove the following:
  a) $(a_n)_n$ is convergent to a iff $\forall\, \epsilon > 0_F\ \exists\, N \in \mathbb{N}$ s.t. for $\forall\, n \succ N$ one has $|a - x_n| \prec \epsilon$.
  b) $(a_n)_n$ is Cauchy iff $\forall\, \epsilon > 0_F\ \exists\, N \in \mathbb{N}$ s.t. for $\forall\, n, m \succ N$ one has $|a_n - a_n| \prec \epsilon$.

That is, in the 2nd part of the definitions one can use $\leqslant$ or $<$, and the math content is the same.

2) For $(a_n)_n \in \mathcal{S}(F)$, negate the following assertions using quantifiers and the definitions:
  a) (a_n) is bounded below, respectively above, respectively bounded.
  b) $(a_n)_n$ is a convergent sequence.
  c) $(a_n)_n$ is a Cauchy sequence.

3) Prove/disprove the following assertions (some from the class):
  a) $(a_n)_n, (b_n)_n \in \mathcal{S}_{\mathrm{b}}(F)$ iff $a^2 \cdot (a_n)_n - b^2 \cdot (b_n)_n, (a_n \cdot b_n)_n \in \mathcal{S}_{\mathrm{b}}(F)$ for all $a, b \in F$.
  b) $(a_n)_n, (b_n)_n, (c_n)_n \in \mathcal{S}_{\mathrm{c}}(F)$ iff $(c_n^2)_n, 2(a_n)_n - 3(b_n)_n, (a_n)_n + (b_n)_n \in \mathcal{S}_{\mathrm{c}}(F)$. Limits?

Formulate/prove/disprove the corresponding assertion a), b) for Cauchy sequences.

4) Prove the so called ***Squeeze Theorem*** (Google it!):
Let $(a_n)_n, (b_n)_n, (x_n)_n \in \mathcal{S}(F)$ be given such that $a_n \leqslant x_n \leqslant b_n$ for $n \gg 0$.
  a) *If $a_n, b_n \to c \in F$, then $x_n \to c$.*
  b) *If $(a_n)_n, (b_n)_n \in \mathcal{S}_{\mathrm{C}}(F)$ and $a_n - b_n \to 0_F$, then $(x_n)_n \in \mathcal{S}_{\mathrm{C}}(F)$, and $x_n - a_n \to 0_F$.*
  c) *If $(a_n)_n, (b_n)_n \in \mathcal{S}_{\mathrm{b}}(F)$, then $(x_n)_n \in \mathcal{S}_{\mathrm{b}}(F)$.*

The case $F = \mathbb{Q}$: Have fun!

5) Answer/prove/disprove the following:

a) For $a \in \mathbb{Q}$, one has: $(a^n)_n \in \mathcal{S}_c(\mathbb{Q})$ iff $(a^n)_n \in \mathcal{S}_C(\mathbb{Q})$ iff $-1 < a \leqslant 1$. Limits?
Is the same true for $a \in F$ an arbitrary totally ordered field?

b) The convergence radius of the geometric series $\sum_n t^n \in \mathbb{Q}[[t]]$ is $\rho = 1$.
Is the same true for $\sum_n t^n \in F[[t]]$ for F an arbitrary totally ordered field?

6) Decimal expansion. Let $(a_n)_n$ be defined by $a_n = m_0 + \frac{m_1}{10^1} + \cdots + \frac{m_n}{10^n}$ for $n \in \mathbb{N}$, where $m_0 \in \mathbb{N}$, and $0 \leqslant m_i < 10$ for all $0 < i \leqslant n$. Prove/disprove/answer the following:

a) $(a_n)_n \in \mathcal{S}_C(\mathbb{Q})$, i.e., $(a_n)_n$ is a Cauchy sequence.

b) $(a_n)_n \in \mathcal{S}_c(\mathbb{Q})$ iff $(m_n)_n$ is almost periodic, i.e., $\exists\, k > 0$ s.t. $m_{n+k} = m_n$ for $n \gg 0$.

[**Hint:** Check "periodic decimal expansion" etc ...]

7) Recurrence sequences. Let $a, b, x_0, x_1 \in \mathbb{Q}$ be given, and for $n > 1$ define inductively: $x_n = a x_{n-1} + b x_{n-2}$. Prove/disprove/answer the following:

a) If $x_0, x_1, a, b > 0$, then $(x_n) \in \mathcal{S}_b(\mathbb{Q})$ iff $a + b \leqslant 1$ iff $(x_n)_n \in \mathcal{S}_C(\mathbb{Q})$.
For which values $a, b, x_0, x_1 > 0$ in $\mathbb{Q}$ is the sequence $(x_n)_n$ convergent in $\mathbb{Q}$?

b) For which values of $a, b, x_0, x_1 \in \mathbb{Q}$ is $(x_n)_n \in \mathcal{S}_c(\mathbb{Q})$, respectively a Cauchy sequence?

[**Hint:** Check "recurrence sequences" and find a *closed formula* for x_n of the form $x_n = \alpha u^n + \beta v^n$, etc ...]

• Supplement: Totally ordered rings/fields in which $\mathbb{Z}$ and/or $\mathbb{Q}$ is not dense.

Let R be a totally ordered domain, e.g. $R = \mathbb{Z}, \mathbb{Q}$, and $R[t]$ be the polynomial ring in the variable t over R. Let $R[t]_0 \subset R[t]$ be the subset of polynomials $p(t) = \sum_n a_n t^n$ with *leading coefficient* $a_{\deg(p)} > 0_R$ together with $0_{R[t]}$.

Example. $-1_{R[t]} \notin R[t]_0$, $-100 + 3t \in R[t]_0$, $10^{200} + 2.8 \cdot 10^{12} t - t^3 \notin R[t]_0$ (WHY).

1) Let $R, +, \cdot, \leqslant$ be a totally ordered domain. In the above notation, prove the following:

a) $R[t]_0 \subset R[t]$ satisfies $-R[t]_0 \cup R[t]_0 = R[t]$, $-R[t]_0 \cap R[t]_0 = \{0_{R[t]}\}$.
Hence $p(t) \leqslant q(t) \overset{\text{def}}{\longleftrightarrow} q(t) - p(t) \in R_0$ makes $R[t]$ into a totally ordered domain (WHY).

b) Let $R = F$ be a field. Then $p(t) \leqslant q(t)$ iff $\exists\, \epsilon = \epsilon_{p,q} > 0_F$ s.t. $\forall\, a > \epsilon$ one has $p(a) \leqslant q(a)$ in F.

Intuitively, $p(t) \leqslant q(t)$ *holds iff* $p(a) \leqslant q(a)$ *holds for* $a \gg 0_F$.

Finally, F is not dense in $F[t]$, i.e., there are $p, q \in F[t]$ s.t. $p < q$ and $(p, q) \cap F = \emptyset$. Give examples of such $p < q$.

[**Hint** to b): Let $r(r) = q(t) - p(t)$. Check the cases $r(t) = 0_{F[t]}$ and $r(t) = a_0$ directly. For $r(t) = a_0 + \cdots + a_N t^N$, $N = \deg(r) > 0$, one has: (i) $r(t) > 0_{F[t]}$ iff $a_N > 0_F$. (ii) If $c > |a_0|, \ldots, |a_{N-1}|$, and $\epsilon > 1_F$, then $|a_0 + \cdots + a_{N-1}\epsilon^N| \leqslant |a_0| + \cdots + |a_{N_1}| \cdot \epsilon^n < Nc \cdot \epsilon^{N-1}$ (WHY). Choose $\epsilon > 1_F$ s.t. $Nc \cdot \epsilon^{N_1} < |a_N|\epsilon^N$, or equivalently, $Nc/|a_N| < \epsilon$. Then $a > \epsilon$ implies: If $a_N < 0_F$, then $p(a) \leqslant a_N a^N + \sum_i |a_i| a^i < 0_F$ (WHY); if $a_N > 0_F$, then $p(a) \geqslant a_N a^n - \sum_i |a_i| a^i > 0_F$ (WHY).