Due: Fr, Oct 22, 2021

Math 202 / Problem Set 6 (one page)

The monoid/group/ring of functions

Let X, T be non-empty sets, and $\mathrm{Maps}(X,T) := \{f \mid f : X \to T \text{ abstract map}\}$. Given a composition law $\cdot$ on T, define the $\bullet$ operation on $\mathrm{Maps}(X,T)$ by $(f \bullet g)(x) := f(x) \cdot g(x)$.

1) Prove/disprove the following assertions from the class:

a) $\bullet$ is associative, reps. commutative iff $\cdot$ is so.

b) $\bullet$ has a neutral element e iff $\cdot$ has a neutral element $e_\bullet$. What is $e_\bullet$ as a function?

c) $f \in \mathrm{Maps}(X,T)$ is invertible w.r.t. $\bullet$ iff $f(x) \in T$ is invertible w.r.t. $\cdot$ for all $x \in X$.

2) In the context of Problem 5) above, prove/answer the assertion form the class:

a) $G, \cdot$ is an (abelian) monoid, reps. group iff $\mathrm{Maps}(X,G), \bullet$ is so.

b) $R, +, \cdot$ is a (commutative) ring with 1_R iff the corresponding $\mathrm{Maps}(X,R), +, \bullet$ is so.

($\bullet$) **Question:** Is the same true for (skew) fields $F, +, \cdot$?

Language: $\mathrm{Maps}(X,T)$ is called the monoid/group/ring of T-valued maps on X.

Products of monoids/groups/rings/(skew) fields

Let $*_1$ and $*_1$ be composition laws on x_1, respectively x_2. Define the coordinate wise composition law $* := *_1 \times *_2$ on $X := X_1 \times X_2$ by $(x_1, x_2) * (y_1, y_2) := (x_1 *_1 y_1,\ x_2 *_2 y_2)$.

3) Prove/disprove/answer the assertions from the class:

a) $*$ is associative, reps. commutative if and only if $*_1$ and $*_2$ are so.

b) $*$ has a neutral element e iff $*_1$ and $*_2$ have neutral elements e_1, e_2.

c) $x := (x_1, x_2)$ is invertible iff x_1 and x_2 are invertible.

4) Let $G := G_1 \times G_2$, $R := R_1 \times R_2$ endowed with the component wise composition laws. Prove the assertions from the class

1) *G_1 and G_2 are (abelian) monoids, resp. groups, iff G is so.*

2) *R_1 and R_2 are (commutative) rings iff R is so.*

($\bullet$) **Question:** Is the same true for fields F_1, F_2 ?

Rings of formal series/formal power series/polynomials

Recall that for a commutative ring $R, +, \cdot$ we defined the rings $\mathbf{\Sigma}(R) = \{\sum_n a_n \mid a_n \in R\}$ of formal series over R, and the rings of formal power series $R[[t]] = \{\sum_n a_n t^n \mid a_n \in R\}$ and the ring of polynomials $R[t] \subset R[[t]]$ in the variable t over R.

5) Compute the inverses w.r.t. multiplication of the following formal (power) series:

a) $\sum_n (-1)^n$; $\sum_n 1$; $\sum_n \frac{1}{2^n}$; $\sum_n 2^n$; $1 + a_1 + \sum_{n>1} 0$ as formal series in $\mathbf{\Sigma}(\mathbb{Q})$.

b) $\sum_n (-1)^n t^n$; $\sum_n t^n$; $\sum_n \frac{1}{2^n} t^n$; $\sum_n 2^n t^n$ as formal power series $\mathbb{Q}[[t]]$.

6) Let $R[t]$ be the polynomial ring over R. For $p(t) = a_n t^n + \cdots + a_0 \in R[t]$, prove/disprove:

a) $p(t)$ is invertible in $R[t]$ iff a_0 is invertible in R and $a_i = 0_R\ \forall\, i > 0$.

b) $\exists\, N > 0$ s.t. $p(t)^N = 0_{R[t]}$ iff $\exists\, N' > 0$ s.t. $a_i^{N'} = 0_R\ \forall\, i$.