Due: Mo, Oct 18, 2021

Math 202 / Problem Set 5 (one page)

1) Show that one has division with remainder in $\mathbb{Z}$, i.e., one has the following:

For $a, b \in \mathbb{Z}$, $b \neq 0$ *there are unique* $q \in \mathbb{Z}$, $r \in \mathbb{N}$ *s.t.* $a = b \cdot q + r, \quad 0 \leqslant r < |b|$.

Composition laws & Basic algebraic structures

2) Let X be a non-empty set, and recall the symmetric difference $A \vartriangle B := (A \backslash B) \cup (B \backslash A)$ on $\mathcal{P}(X)$. Prove/disprove the following:

a) The difference $A \backslash B$ on $\mathcal{P}(X)$ is not associative/commutative/has no neutral element.

b) $\mathcal{P}(X), \vartriangle$ is an abelian group.

3) In the notation from Problem 2) above, prove/answer the following:

a) $\mathcal{P}(X), \vartriangle, \cap$ is a commutative ring.

b) Which elements in the ring $\mathcal{P}(X), \vartriangle, \cap$ are invertible?

4) Prove/answer the following:

a) There exist $\sigma, \tau \in S_3$ such that $(\sigma \cdot \tau)^2 \neq \sigma^2 \cdot \tau^2$.

b) Solve the equations $x \circ \binom{1\,2\,3}{2\,3\,1} = \binom{1\,2\,3}{2\,1\,3}$ and $\binom{1\,2\,3}{2\,3\,1} \circ x = \binom{1\,2\,3}{2\,1\,3}$ in S_3.

c) Find the smallest $n_G > 0$ s.t. $\sigma^{n_G} = e_G$ for all $\sigma \in G$, where: i) $G = S_3$; ii) $G = S_5$.

5) Denote i) $A_1A_2A_3$ triangles; ii) $B_1B_2B_3B_4$ parallelograms; iii) $C_1C_2C_3C_4C_5$ pentagons with three equal sides $|C_1C_2| = |C_2C_3| = |C_3C_4|$. Depending of further properties of these shapes, write in each case the group of transformations as permutation groups of the vertices, hence the results will be subgroups of S_3, S_4, S_5, respectively (WHY). [**Note** that the groups of transformations depend on the geometric properties of the shapes under discussion; e.g. in the case of triangles $A_1A_2A_3$, the group can be $\left\{\binom{1\,2\,3}{1\,2\,3}\right\}$, $\left\{\binom{1\,2\,3}{1\,2\,3}, \binom{1\,2\,3}{1\,3\,2}\right\}$, or S_3 (WHY), etc.]

6) For a commutative ring $R, +, \cdot$ and $a, b, c, d \in R$, using $+$ and $\cdot$ define on R a new "addition" by $x \boldsymbol{+} y = x + y + a$ and a new "multiplication" by $x \bullet y = xy + bx + cy + d$.

a) Find all a, b, c, d such that $R, \boldsymbol{+}, \bullet$ is a ring, in the cases: i) $R = \mathbb{Z}$; ii) $R = \mathbb{Q}$.

b) Solve in the ring $R, \boldsymbol{+}, \bullet$ the equations $x^2 \boldsymbol{+} 3 \bullet x = 1_R$ and $x^2 \boldsymbol{+} 3 \bullet x = 0_R$.

The quaternions $\mathbb{H}_R$ attached to a commutative ring R

Let R be a commutative ring with $1_R \neq 0_R$, and define the *quaternions* $\mathbb{H}_R$ *over* R by $\mathbb{H}_R := R^4 := \{a + b\,\imath + c\,\jmath + d\,\kappa \mid a, b, c, d \in R\}$ endowed with the coordinate wise addition $+$ and the multiplication $\cdot$ defined by: $\imath^2 = \jmath^2 = \kappa^2 = -1_R$, $\imath \cdot \jmath = \kappa$, $\jmath \cdot \kappa = \imath$, $\kappa \cdot \imath = \jmath$.

7) Prove the following:

a) $\mathbb{H} := \mathbb{H}_{\mathbb{Q}}$ is a skew field. And solve the equation $(1 + \imath + \jmath + \kappa) \cdot x = \jmath + x \cdot \imath$.

b) Show that $\mathbb{H}_R, +, \cdot$ is a ring with $1_{\mathbb{H}_R}$, and $\mathbb{H}_R, +, \cdot$ is commutative iff $1_R = -1_R$.

c) The map $\phi : R \to \mathbb{H}_R$, $a \mapsto a + 0\,\imath + 0\,\jmath + 0\,\kappa$, is compatible with the addition and multiplication, i.e., $\phi(a + b) = \phi(a) + \phi(b)$, $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ $\forall\, a, b \in R$.