Due: Thr, Sept 30, 2021

Math 202 / Problem Set 3 (two pages)

Basics: Logical deduction, Sets, Functions, . . .

1) Recall that given a degree two equation $ax^2 + bx + c = 0$ with real number coefficients $a \neq 0, b, c$, the number $\Delta := b^2 - 4ac$ is called the discriminant. Answer the following:

a) Write the following assertions using quantifiers:

- *A degree two equation with real coefficients has real roots iff its discriminant is non-negative.*
- *A degree two equation with real coefficients has two distinct real roots iff its discriminant is positive.*

b) Write down the negations of the above assertions, both as assertions in plain English, and using quantifiers.

(∗) Are the above assertions true ?

2) Let $f, g : (a, b) \to \mathbb{R}$ be functions. Recall that f is continuous at a point $x_0 \in (a, b)$, if f satisfies: $\forall\, \epsilon > 0\ \exists\, \delta > 0$ s.t. $\Big(|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon\Big)$. Answer the following:

a) Using quantifiers, negate the fact that f is continuous at x_0.

b) Using the definition of continuity at x_0, negate the following assertion:
If the sum of f and g is continuous at x_0, then both f and g are continuous at x_0.

Cardinality of sets. (See/study **Notes**, p.8-9, e.g. **Thm** of Cantor, Bernstein, Schroeder).

• Recall the definitions (see **Notes**, Def 1.44): $[0] = \emptyset$, $[n] = \{1, \dots, n\}$ for $n \neq 0$, and that a set X has cardinality $|X| = n$, resp. countable, resp. at most countable, resp. uncountale.

3) Let X, Y be finite sets, say $|X| = m$ and $|Y| = n$. Prove/disprove the following assertions:

a) $|X \cup Y| + |X \cap Y| = |X| + |Y|$. What is the corresponding assertion for $|X \cup Y \cup Z|$?

b) $|X \times Y| = |X| \cdot |Y|$. What is the corresponding assertion for $|X \times Y \times Z|$?

4) Let A, B be finite sets, and $m := |A|$, $n := |B|$ be their cardinalities. Prove/answer:

a) $\mathrm{Maps}(A, B)$ has cardinality $|\mathrm{Maps}(A, B)| = n^m$.

b) What are the cardinalities of $\mathrm{Inj}(A, B), \mathrm{Srj}(A, B), \mathrm{Bij}(A, B)$? (. . . last resort: Google it!)

5) Let A, B, A_n, $n \in \mathbb{N}$ be at most countable sets. Prove the following assertions:

a) $A \times B$ is at most countable. Is the same true for $A_0 \times \dots \times A_n$ for all $n \in \mathbb{N}$?

b) $A \cup B$ is at most countable. Is the same true for $\cup_{n \in \mathbb{N}} A_n$?

c)* Is the same true for the (infinite cartesian) product $A_0 \times \dots \times A_n \times \dots$?

6) Let X be an arbitrary set, and $\mathcal{P}(X) := \{A \mid A \subseteq X\}$ be the power set of X. Prove:

a) If $|X| = n$ is finite, then $|\mathcal{P}(X)| = 2^n$.

b) One has always: $|X| < |\mathcal{P}(X)|$ (Google it!). Deduce from this that $|\mathbb{N}| < |\mathbb{R}|$.

[**Hint** to the second part of b): Define $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by $f(A) := \overline{a_0 \,.\, a_1 a_2 \dots a_n \dots}$ for $A \subset N$, where $a_n = 1$ if $n \in A$, and $a_n = 0$ if $n \notin A$. Then $A \neq A'$ implies $x_A \neq x_{A'}$ (WHY), hence f is injective, etc. . .]

More about order relations

Recall the for a (partially) ordered set $A, \leqslant$, and a non-empty subset $B \subset A$, one defines $\min(B)$, $\max(B)$, $\inf(B)$, $\sup(B)$ **—if they exist(!)**

7) Let $\mathbb{R}$ be the set of real numbers endowed with the usual ordering $\leqslant$ (as you know them in the naive sense). [**NOTE**: We will defined all these notions in a rigorous way later!]
Let $B \subset A$ be sets of real numbers. Find $\min(B)$ $\max(B)$, $\inf(B)$, $\sup(B) \in A$, provided they exist, in the following cases:

a) $A = \mathbb{Q}$ is the set of rational numbers, and each of the case below (considered separatly):
(i) $B = \{a \in X \,|\, -1 \leqslant a^2 \leqslant 4\}$; (ii) $B = \{a \in X \,|\, 1 \leqslant a^2 \leqslant 3\}$; (iii) $B = \{a \in X \,|\, 2 \leqslant a^3\}$

b) Same questions, but for $A = \mathbb{Q} \cup (1, \infty)$, the union of $\mathbb{Q}$ and the open interval $(1, \infty)$.

8) Let $A, \leqslant$ be a partially ordered set, $B \subset A$ non-empty subset. Prove/disprove/answer:

a) $\min(B)$ exists iff $\inf(B)$ in A exists and lies in B. Same question for $\max(B)$.

b) If $\inf(B)$ exists for all B, then $\leqslant$ is a total ordering. Is the converse true?

c) If $\leqslant$ is a total ordering on A, then $\inf(B)$ exists for all B.

9) Let $X \neq \emptyset$ be a set, $\mathcal{P}(X)$ is power set, and a map $f : X \to X$, recall the maps

$$f^*,\ f_* : \mathcal{P}(X) \to \mathcal{P}(X) \text{ by } f_*(X') := f(X'),\ f^*(X') := f^{-1}(X') \text{ for } X' \in \mathcal{P}(X).$$

As mentioned in class, define the relation $\leqslant$ on $\mathcal{P}(X)$ by $X' \leqslant X'' \overset{\text{Def}}{\Longleftrightarrow} X' \subset X''$. Prove that $\leqslant$ is a partial ordering on $\mathcal{P}(X)$, and prove/disprove/answer the following:

a) If $\mathcal{A} \subset \mathcal{P}(X)$, then $\inf(\mathcal{A}) = \cap_{X' \in \mathcal{A}} X'$ and $\sup(\mathcal{A}') = \cup_{X' \in \mathcal{A}} X'$.

b) Is $f_* : \mathcal{P}(X) \to \mathcal{P}(X)$ is compatible with $\leqslant$, that is, $X' \leqslant X'' \Rightarrow f_*(X') \leqslant f_*(X'')$.

c) The same question for $f^* : \mathcal{P}(X) \to \mathcal{P}(X)$.

10) Let A be a finite set. Prove/Answer the following (some/most of it done in the class):

a) An ordering $\leqslant$ on A is a well ordering iff it is a total ordering.

b) Let $\leqslant, \leqslant'$ be any two totally orderings of A. Then there exists a unique bijection $f : A \to A$ satisfying: $\forall\, x, y \in A$ one has $x \leqslant y$ iff $f(x) \leqslant' f(y)$.

c) What is the converse of the above assertion, and is the converse true?

d) How many total orderings of A are there?

11) Recall that for $m, n \in \mathbb{N}$, one defines: $m \leqslant n \overset{\text{Def}}{\Longleftrightarrow} \exists\, l \in \mathbb{N}$ such that $n = m + l$. Complete the proof of the assertions from the class that $\leqslant$ is compatible with $+$ and $\cdot$ and has the cancelation property, i.e., $\forall, n, m, k \in \mathbb{N}$ one has:

(i) $m \leqslant n$ iff $m + k \leqslant n + k$. (ii) $m \leqslant n$ iff $m \cdot k \leqslant n \cdot k$, provided $k > 0$.

12) Complete the proof of the fact: *The natural ordering $\leqslant$ on $\mathbb{N}$ is a well ordering.*

[**Hint**: For $n \in A \subset \mathbb{N}$, let $A_{\leqslant n} := \{m \in A \,|\, m \leqslant n\}$. Then $\min(A_{\leqslant n})$ exists by **10)**, a) above, and $\min(A) = \min(A_{\leqslant n})$ (WHY).