Due: Thr, Sept 23, 2021

Math 202 / Problem Set 2 (two pages)

Basics of logical deduction

1) Consider the assertions in plain English: $p \equiv$ (all doors will open), $q \equiv$ (the train stops). Answer the following:

a) What is the (logical) negation in plain English of the assertion p, respectively q.

b) Using $\neg$, $\&$, $\vee$ and p, q, write down the following assertion:
"Either all doors will open, or the train does not stop."

2) Let p, q be the assertions $p \equiv$ (more jobs), $q \equiv$ (lower taxes), $r \equiv$ (increase spending). In country X, statistical data show that lately more jobs were created. Explain what is logically faulty with the following assertions —dear to some politicians/economists/others:

a) "You see, since we lowered taxes, more jobs were created."

b) "You see, because we did not lower the taxes, we could increase spending, and therefore more jobs were created."

[**Hint:** Write down the assertion a), b) as logical assertions using p, q, r, and see whether the cause-effect is the one explained by politicians/economists/some; recall that "wrong implies everything"...]

Working with Sets and Functions/Maps

3) Let $A, B, C, A_i, i \in I$ be sets. Prove the following properties of (direct) product of sets:

a) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ and $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

b) $(\cup_i A_i) \times C = \cup_i (A_i \times C)$ and $(\cap_i A_i) \times C = \cap_i (A_i \times C)$.

Language/Terminology: The product of sets is distributive w.r.t. union and intersection.

● Make sure that you check the details for the proofs of the assertions from the class:

Let A, B be arbitrary sets. The in notations as introduced in class, the following hold:

- If $R \subset A \times B$ subset, then $R^{-1} :\stackrel{\text{def}}{=} \{\, (y, x) \mid (x, y) \in R \,\}$ is subset of $B \times A$.
- If $R \subset A \times B$, then $pr_B(R) :\stackrel{\text{def}}{=} \{\, y \mid \exists\, x \in A \text{ s.t. } (x, y) \in R \,\}$ is subset of B.
- If $R \subset A \times B$, $S \subset B \times C$ are correspondences, then $S \circ R \subset A \times C$ is a correspondence.
- If $R \subset A \times B$ is funct. corresp., then $f_R : A \to B$ as defined in class (HOW) is a function.
- If $f : A \to B$ is a map, then $R_f \subset A \times B$ as defined in class (HOW) is a funct. corresp.
- In the above cases one has $f_{R_f} = f$ and $R_{f_R} = R$. (How is the equality of maps defined?)
- If $T \subset A \times B$, $S \subset B \times C$ are funct. corresp., then so is $S \circ R$, and $f_{S \circ R} = f_S \circ f_T$.
- If $f : A \to B$, $g : B \to C$ are maps, then $R_{g \circ f} = R_g \circ R_f$.

4) Let A, B, C, D be sets. Prove/disprove the following:

a) $A \times B \to B \times A$, $(x, y) \mapsto (y, x)$ defines a bijective map.

b) $(A \times B) \times C \to A \times (B \times C)$, $((x, y), z) \mapsto (x, (y, z))$ defines a bijective map.

5) Let $f_1 : A_1 \to B_1$, $f_2 : A_2 \to B_2$ be maps. Prove/disprove/answer the following:

a) $f : A_1 \times A_1 \to B_1 \times B_2$, $(x_1, x_2) \mapsto \big(f_1(x_1), f_2(x_2)\big)$ is a map.

b) The map f is injective/surjective/bijective iff f_1, f_2 are so.

6) Let $f : A \to B$, $g : B \to C$ be maps, hence $g \circ f : A \to C$. Prove/disprove the following:

a) If f and g are injective (resp. surjective), then $g \circ f$ is injective (resp. surjective).

(?) If $g \circ f$ is injective (resp. surjective), then that f and g are injective (resp. surjective).

b) If f and g are bijective, then $g \circ f$ is bijective. Is the converse assertion true?

7) Let A, B be sets. Answer the following:

a) There is an injective map $f : A \to B$ iff there exists a surjective map $g : B \to A$.

b) $f : A \to B$ is injective iff there is $g : B \to A$ surjective s.t. $g\big(f(x)\big) = x \;\forall\, x \in A$.

c) $f : A \to B$ is surjective iff there is $g : B \to A$ injective s.t. $f\big(g(y)\big) = y \;\forall\, y \in B$.

Definition. Let $f : A \to B$ be an arbitrary map. For subsets $A' \subset A$ and $B' \subset B$ define:

a) $f(A') := \{y \in B \,|\, \exists\, x \in A' \text{ with } f(x) = y\}$, called the image of A' under f.

b) $f^{-1}(B') := \{x \in X \mid f(x) \in B'\}$, called the pre-image of B' under f.

Recall that $f(A') \subset B$ and $f^{-1}(B') \subset A$ are indeed subsets (**Notes**, Prop. 1.37).

8) Prove/disprove that for all subsets $A', A'' \subset X$ and $B', B'' \subset Y$ one has:

a) $f(A' \cap A'') = f(A') \cap f(A'')$, respectively $f(A' \cup A'') = f(A') \cup f(A'')$.

b) $f^{-1}(B' \cap B'') = f^{-1}(B') \cap f^{-1}(B'')$, respectively $f^{-1}(B' \cup B'') = f^{-1}(B') \cup f^{-1}(B'')$.

(•) The same questions provided f is injective, resp. surjective, resp. bijective.

9) Let A, B be sets, and $\mathrm{Inj}(A, B), \mathrm{Srj}(A, B), \mathrm{Bij}(A, B), \mathrm{Maps}(A, B)$ be the collections of all the injective, resp. surjective, resp. bijective, resp. arbitrary maps $f : A \to B$.
Using the ZF Axioms, prove that $\mathrm{Inj}(A, B), \mathrm{Srj}(A, B), \mathrm{Bij}(A, B), \mathrm{Maps}(A, B)$ are sets.

- Learn/study the proofs of Theorems 1.38, 1.40, 1.41, 1.49 from the Notes
(concerning $\mathbb{N}$, the Induction Principle, cardinality of sets, and finite sets).

More about the set of natural numbers $\mathbb{N} = \{\, 0, 1, 2, \ldots \}$

10) Complete the proof of the assertions from the class:

a) $+$ and $\cdot$ in $\mathbb{N}$ are associative, commutative, have neutral elements, and $\cdot$ is distributive w.r.t to $+$

b) $+$ and $\cdot$ have cancelation property in $\mathbb{N}$, respectively $\mathbb{N}_{>0}$, i.e., for $n, m, k \in \mathbb{N}$ one has:

(i) $m + k = n + k \Rightarrow m = n$. (ii) $m \cdot k = n \cdot k \Rightarrow m = n$, provided $k \neq 0$.

11) Let $m_1, \ldots, m_r, n_1, \ldots, n_s \in \mathbb{N}$ given. Prove/answer the following:

a) The value of $m_1 + \cdots + m_r$ and $m_1 \cdots m_r$ are well defined, i.e., their value does not depend on the order in which one performs the additions, respectively the multiplications.

b) One has the following: $(m_1 + \cdots + m_r)(n_1 + \cdots + n_s) = \sum_{i,j} m_i n_j$.