Due: **Optional** / Fr, Dec 10, 2021

Math 202 / Problem Set 12 (two pages)

Unform continuity.

1) Let $f : X \to Y$ be cont map of metric spaces, $y_0 \in Y$ fixed. Prove the following:

a) $g : X \to \mathbb{R}$, $x \mapsto d\big(f(x), y_0\big)$ is a continuous map.

b) If X is compact, then f is *uniformly continuous,* i.e., the following holds:

$$\forall\, \epsilon > 0\ \exists, \delta > 0 \text{ s.t. } d(x, x') < \delta \Rightarrow d\big(f(x), f(x')\big) < \epsilon.$$

Differentiability. Recall that either $F = \mathbb{R}$ and $\emptyset \neq U \subset \mathbb{R}$ is open, or $\mathbb{C}$ and $\emptyset \neq U \subset \mathbb{C}$ is open. Let $f, g : U \to F$ be differentiable at $x_0 \in U$. Make sure that you checked all the details of the assertions from the class:

- f, g are continuos at x_0.
- $(f+g)'(x_0) = f'(x_0) + g'(x_0)$, $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
- If $g(x) \neq 0$ on U, then $(f/g)'(x_0) = \frac{1}{g^2(x_0)}\Big(f'(x_0)g(x_0) - f(x_0)g'(x_0)\Big)$.

2) Using the chain rule, prove the following:

Let $f : U \to V$ and $g : V \to U$ be inverse to each other, and suppose that $f'(x_0)$ exists and $f'(x_0) \neq 0$. Then setting $y_0 = f(x_0)$, one has: $g'(y_0)$ exists, and $f'(x_0)g'(y_0) = 1$.

3) For $a < b$ in $\mathbb{R}$, let $f : [a, b] \to \mathbb{R}$ be a continuous map, differentiable on (a, b). Prove/study/answer the following basic facts:

a) f is non-monotone iff $f'(x)$ takes strictly positive and strictly negative values.

b) Suppose that $f'(x)$ takes both strictly positive and strictly negative values. Does $f'(x)$ take value 0 as well, i.e., is there $x_0 \in (a, b)$ s.t. $f'(x_0) = 0$?

Basics about diff functions

4) Let $I := (a, b)$ and $f : I \to \mathbb{R}$ be differentiable on I s.t. $f'(x) \neq 0\ \forall\, x \in I$. Prove:

a) $f(I) = J$ is an open interval, and $f : I \to J$ is bijective.

b) $f^{-1} : J \to I$ is differentiable, and $(f^{-1})'(y) = 1/f'(x)$, where $f(x) = y$.

[**Hint**: First (a, b) is connected, and f is continuous (WHY), etc... $f'(x) \neq 0$ for all x implies f strictly monotone (WHY), etc.]

5) Let $u : U \to (0, \infty)$ and $v : U \to \mathbb{R}$ be differentiable. Prove/answer the following:

a) What are the derivatives $(u^\alpha)'$, $(a^v)'$, $(\log_a u)'$ for $a \in \mathbb{R}_{>0}$, $a \neq 1$?

b) The map $u^v : U \to (0, \infty)$ defined by $u^v(x) :\overset{\text{def}}{=} u(x)^{v(x)}$ is differentiable. Compute $(u^v)'$.

More about $\sin(x)$ and $\cos(x)$

6) Prove in detail the assertions from the class:

a) $\cos(0) = 1$, $\cos(2) < 0$. Hence $X_0 := \{x \,|\, 0 < x < 2 \ \cos(x) = 0\}$ has $\theta := \inf(X) \in \mathbb{R}$.

b) $\sin(\theta) = 1$, $\cos(2\theta) = -1, \sin(2\theta) = 0$, $\cos(3\theta) = 0$, $\sin(3\theta) = -1\ \cos(4\theta) = 1\ \sin(4\theta) = 0$.

c) The analytic functions $\sin(x)$ and $\cos(x)$ are periodic with period 4θ.

[**Hint**: To b) and c): Use the addition formulas ...]

Notation/Remark. $\pi :\overset{\text{def}}{=} 2\theta$ is the famous number discovered by the ancient Greeks, relating to *squaring the circle.* π is *transcendental*, and using computation methods one gets

$\pi = 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812$

The geometric point of view for $\sin(x)$ and $\cos(x)$

Recall the definitions of the following notions (corresponding to physical intuition):

- The lines in $\mathbb{R}^2 = \mathbb{C}$ are $L_{a,b,c} := \{P = (x,y) \mid ax + by = c\}$ for fixed $a, b, c \in \mathbb{R}$.
- The circle of center $P \in \mathbb{R}^2$ and radius $r > 0$ is $CrP := \{P' \mid d(P, P') = r\}$.
- The segment $[P,Q]$ is $[P,Q] :\overset{\text{def}}{=} \{R \in \mathbb{R}^2 \mid d(P,R) + d(R,Q) = d(P,Q)\}$.
- For every $P \neq Q$ there exists a unique line L_{PQ} with $P, Q \in L$.
- If $P \in [P,Q]$, then P' lies on the line L_{PQ}. What is the converse of this?
- Let $P' = (a,b) \in CrP$ be given. Then for $P'' \in CrP$, TFAE:

  (i)] $P'' := (x_P - a, y_P - b)$; (ii) $P \in [P', P'']$; (iii) $d(P', P'') = 2r$.

  In particular, $P'' \in CrP$ satisfying (i), (ii), (iii) is unique.

Definition. Given P', $P'' \in CrP$ as above, $[P', P'']$ is called a *diameter* of CrP.

- Conclude that $\mathcal{C} :\overset{\text{def}}{=} C1O = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} = \{P \in \mathbb{R}^2 \mid d(P,O) = 1\}$, has diameters of the form $[P', P'']$ with $P' = (a,b) \in \mathcal{C}$ and $P'' := (-a, -b)$, $a^2 + b^2 = 1$.

1) Let $I := [-1, 1]$, and define two maps $\gamma^\pm : I \to \mathcal{C}$ by $\gamma^\pm(t) = (t, \pm\sqrt{1 - t^2}\,)$. Prove:

a) γ^+ is continuous and bijective onto the *upper semicircle* $\mathcal{C}^+ = \{(x,y) \in \mathcal{C} \mid y \geqslant 0\}$ of $\mathcal{C}$.

b) γ^- is continuous and bijective onto the *lower semicircle* $\mathcal{C}^- = \{(x,y) \in \mathcal{C} \mid y \leqslant 0\}$ of $\mathcal{C}$.

2) Let $(a,b) \in \mathcal{C}$, and define $\varphi : \mathbb{R}^2 \to \mathbb{P}^2$ by $\varphi(x,y) = (ax - by, bx + ay)$. Prove that φ is a *Euclidean transformation* of $\mathbb{R}^2$, i.e., preserves distances, and maps lines onto lines and circles onto circles of the same radius. In particular, if $\gamma : I \to \mathbb{R}^2$ is a path whose length $\ell(\gamma)$ is defined (WHAT'S THAT?), then $\gamma_\varphi := \varphi \circ \gamma : I \to \mathbb{R}^2$ is a path with length $\ell(\gamma_\varphi) = \ell(\gamma)$ (WHY).

Q: Is the same true, correspondingly, for the area $\mathcal{A}(X)$ (IF DEFINED) of subsets $X \subset \mathbb{R}^2$?

3) In the above notation, let $O' := (1,0)$, $O'' := (-1,0)$, $P' := (a,b)$, $P'' := (-a,-b)$. Prove:

a) $\varphi(O') = P'$, $\varphi(O'') = P''$, and φ maps $[O', O'']$ onto $[P', P'']$.

b) If $P, Q \in \mathcal{C}^+$ are any two points, then the length of the circle arc $\mathcal{C}_{PQ}$ is defined (WHY).

c) $\varphi(\mathcal{C}_{PQ})$ is the circle arc with endpoints $\varphi(P), \varphi(Q)$, and $\ell(\mathcal{C}_{PQ}) = \ell\Big(\varphi(\mathcal{C}_{PQ})\Big)$.

Remark/Definition. Let $\pi \in \mathbb{R}$ denote the length of the semicircle $\mathcal{C}^+$. Then starting with $(a,b) := (-1,0)$, it follows that $\varphi(\mathcal{C}^+) = \mathcal{C}^-$ and therefore, $\ell(\mathcal{C}^-) = \pi = \ell(\mathcal{C}^+)$ (WHY).

4) In the above notation, prove the following:

a) For $0 \leqslant t \leqslant \pi$ there exists a unique point $P_t \in \mathcal{C}^+$ such that $\ell(\mathcal{C}^+_{O'P_t}) = t$.
For $-\pi \leqslant t' \leqslant 0$ there exists a unique point $P_{t'} \in \mathcal{C}^-$ such that $-\ell(\mathcal{C}^-_{O'P'_t}) = t'$.

b) There exists a *unique map* $\theta : \mathbb{R} \to \mathcal{C}$, say $\theta(u) =: P_u =: (x_{P_u}, y_{P_u})$, satisfying:

(i) $\theta(t) = P_t$ for $0 \leqslant t \leqslant \pi$. (ii) $\theta(t') = P_{t'}$ for $-\pi \leqslant t' \leqslant 0$. (iii) $\theta(u + 2\pi) = \theta(u)$ for all $u \in \mathbb{R}$.

Definition. Define the (geometric) $\mathbf{sin}, \mathbf{cos} : \mathbb{R} \to \mathbb{R}$ by $\mathbf{cos}(u) :\overset{\text{def}}{=} x_{P_u}$, $\mathbf{sin}(u) :\overset{\text{def}}{=} y_{P_u}$).

5) Prove that $\mathbf{sin}, \mathbf{cos} : \mathbb{R} \to \mathbb{R}$ are continuous, differentiable, and $\mathbf{sin}' = \mathbf{cos}$, $\mathbf{cos}' = -\mathbf{sin}$.

a) Prove that the addition formulas from Problem 2) hold for all $x \in \mathbb{R}$ and $|y| < \frac{\pi}{4}$.

b) Finally prove that $\mathbf{sin} = \sin$, $\mathbf{cos} = \cos$.